Problem 3 (25%)

An XOR-OR Boolean formula is constructed from the following rules:

- the Boolean constants 0 (false) and 1 (true),
- Boolean variables: x_1, x_2, ..., x_n,
- the Boolean operators: $\oplus$ (exclusive-or) and $\vee$ (inclusive-or), and
- parenthesis.

The Boolean operator $\oplus$ is the exclusive-or operator, given by the following truth table:

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

For example, the formula

$$\phi = (x_1 \oplus x_2) \oplus (1 \vee x_1)$$

is an XOR-OR formula. The decision problem associated with the satisfiability of XOR-OR Boolean formulas is

$$\text{XOR-OR-SAT} = \{\langle\phi\rangle : \phi \text{ is a satisfiable XOR-OR Boolean formula }\}.$$

For example,

$$\phi = (x_1 \oplus x_2) \oplus (1 \vee x_1)$$

belongs to XOR-OR-SAT since ϕ is satisfiable (for example, for the truth assignment $x_1 = 1, x_2 = 1$).

Question 3.1

Show that XOR-OR-SAT $\in$ NP.

Solution: The certificate consists of a satisfying truth assignment. The verification algorithm replaces each variable with the corresponding truth-value and then evaluates ϕ. If ϕ evaluates to 1, the formula is satisfiable. Clearly, this procedure can be performed in polynomial time.

Question 3.2

Show that XOR-OR-SAT is NP-complete. Hint: you may have to use the identity: $P \wedge Q \Leftrightarrow \neg(\neg P \vee \neg Q)$.

Solution: We reduce 3-CNF-SAT to XOR-OR-SAT. Given a Boolean formula ϕ in 3-CNF, we construct a XOR-OR formula ϕ' such that ϕ is satisfiable if and only if ϕ' is satisfiable. Furthermore, the size of ϕ' is bounded by a polynomial in the size of ϕ.

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We can write ϕ as a conjunction of k clauses:

$$\begin{aligned}\phi &= C_1 \land C_2 \land \cdots \land C_k \\ &= \neg(\neg C_1 \lor \neg C_2 \lor \cdots \lor \neg C_k)\,.\end{aligned}$$

Observe that after this transformation, ϕ only contains negation ($\neg$) and disjunction ($\lor$). ϕ' is obtained from ϕ by replacing all occurrences of $\neg\psi$ by $1 \oplus \psi$. The size of ϕ' is linear in the size of ϕ.

Since 3-CNF-SAT is NP-complete and 3-CNF-SAT $\leq_{\mathrm{P}}$ XOR-OR-SAT, it follows from Lemma 38.6 that XOR-OR-SAT is NP-complete.

Question 3.3

Professor C. Lever claims that he has an algorithm which solves the decision problem for XOR-OR Boolean formulas in time $O(n^3 \log\log 2^{2n})$ where n is the length of the formula. Discuss whether this claim sounds reasonable.

Solution: Professor C. Lever's claims to have a polynomial time algorithm for an NP-complete problem. This is not likely since he would then have proved that NP = P.

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