Efficient Algorithms and Programming

Week 11

Reading before Monday November 12th

CLR Chapter 36.3-4.

Exercises for Monday November 12th

1. In section 36.3: Exercises 36.3-1, 36.3-2.
2. In section 36.4: Exercises 36.4-1, 36.4-5, 36.4-6, and 36.4-7 (this one is difficult!).

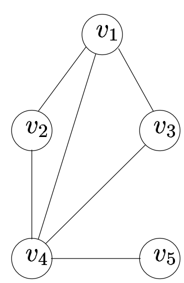
Assignment for Friday November 19th

Problem 1 (E96, 30%)

An *independent set* of a graph $G = (V, E)$ is a subset $V' \subseteq V$ of vertices such that each edge in E is incident on at most one vertex in V'. The *independent-set problem* is to determine a maximum-size independent set in G.

Question 1.1

An independent set of size two for the graph below is $V' = \{v_1, v_5\}$. Find a larger independent set for this graph.



Question 1.2

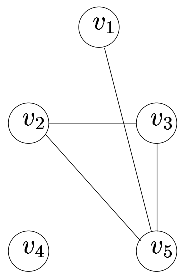
Formulate the related decision problem, INDEPENDENT-SET, for the independent-set problem.

Question 1.3

Show that INDEPENDENT-SET $\in$ NP .

Question 1.4

Given a graph $G = (V, E)$, construct a graph H as the *complement* of G, that is, let H be a graph with vertex set V and edge set $\{(u, v) \mid u, v \in V, u \neq v, \text{ and } (u, v) \notin E\}$. The complement of the above graph is:



Show that G has a clique of size k if and only if H has an independent set of size k.

Question 1.5

Complete the proof that INDEPENDENT-SET is NP-complete.

Question 1.6

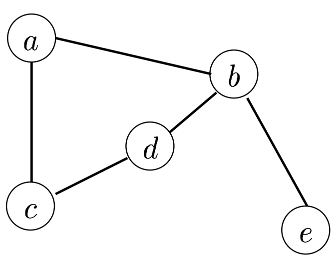
Assume each vertex of G has degree two. Construct a linear time algorithm to find a maximal independent set in G.

Problem 2 (E97, 20%)

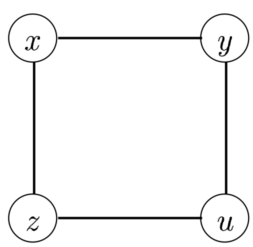
Given two undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the subgraph-isomorphism problem is to determine whether G_1 contains a subgraph isomorphic to G_2, that is, to determine whether there exists a subset $V \subseteq V_1$ and a subset $E \subseteq E_1$ such that $|V| = |V_2|$ and $|E| = |E_2|$ and there exists an injective function $f : V_2 \to V$ satisfying $(u, v) \in E_2$ if and only if $(f(u), f(v)) \in E$.

As an example, consider the two graphs G_1 and G_2 below:

G_1:



G_2:



In this example, G_1 contains subgraphs isomorphic to G_2. One possible subgraph is given by $V = \{a, b, c, d\}$, $E = \{(a, b), (b, d), (d, c), (c, a)\}$, and f given by $f(x) = a, f(y) = b, f(z) = c, f(u) = d$.

The related decision problem, SUBGRAPH-ISOMORPHISM, for the subgraph-isomorphism problem is

$$\text{SUBGRAPH-ISOMORPHISM} = \{\langle G_1, G_2\rangle : G_1 \text{ contains a subgraph isomorphic to } G_2\}\ .$$

Question 2.1

Show that SUBGRAPH-ISOMORPHISM is in NP.

Question 2.2

Show that SUBGRAPH-ISOMORPHISM is NP-hard.

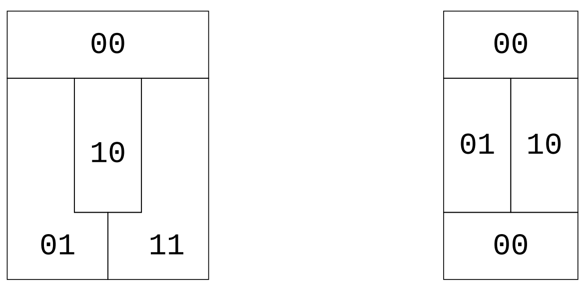
Hint: Consider restricting SUBGRAPH-ISOMORPHISM to cases where G_2 is a complete graph and use that CLIQUE is NP-complete.

Question 2.3

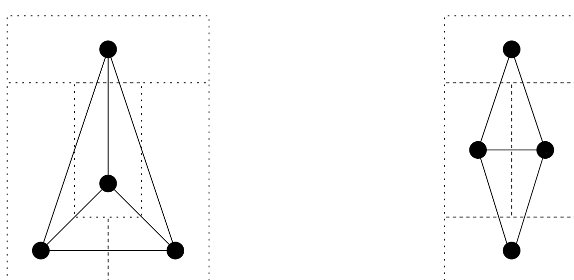
Prove that SUBGRAPH-ISOMORPHISM is NP-complete.

Problem 3 (E98, 40%)

In this problem we consider the colouring of maps of countries. It is a famous theorem that any map of countries can be coloured with only *four* colours so that no two adjacent countries have the same colour. Below are shown two maps. The first requires four colours, the second only three. We use the values $\{00, 01, 10, 11\}$ for the four colours.



For the purpose of finding an assignment of colours to the countries, we can model a map as an undirected graph $G = (V, E)$, where the vertices $V = \{1, \ldots, n\}$ represent the countries, and the edges $E \subseteq V \times V$ represent the neighbouring relation. The two maps above have the following associated graphs:



We take a colouring of a graph G to be a vector $\vec{c} = \langle c_1, \ldots, c_n \rangle$ assigning a colour $c_i \in \{00, 01, 10, 11\}$ to each of the vertices. A *valid colouring* is a colouring $\vec{c}$ of G such that

$$\forall (i, j) \in E.\ c_i \neq c_j\ .$$

ROBDDs can be used to find valid colourings. For each country i, we use two Boolean variables (x_i, y_i) representing the colour of that country. For instance, if $c_i = 01$ then $(x_i, y_i) = (0, 1)$.

Question 3.1

Write down an expression over the Boolean variables expressing the condition that two given countries i and j have different colours. Draw an ROBDD corresponding to this expression.

Question 3.2

Write down a Boolean expression over the variables $x_1, y_1, \ldots, x_n, y_n$ expressing that the variables represent a valid colouring of a graph G.

Question 3.3

Describe an algorithm that efficiently finds a colouring and counts the number of possible different colourings in an undirected graph.

Another problem associated with colourings is to determine whether three colours is enough, i.e., whether there exists a valid colouring using only three colours. The decision problem can be formulated as follows:

$$\text{3-COL} = \{\langle G \rangle \mid G \text{ admits a valid colouring } \vec{c} \text{ with only three colours}\}\ .$$

Question 3.4

Show that 3-COL is in NP.

Professor C. Lever has shown the following three polynomial-time reductions:

- CIRCUIT-SAT $\leq_{\mathrm{P}}$ 3-COL
- 3-COL $\leq_{\mathrm{P}}$ 3-CNF-SAT
- 2-CNF-SAT $\leq_{\mathrm{P}}$ 3-COL

Question 3.5

Use one of his reductions to show that 3-COL is NP-complete.

Question 3.6

Describe how to use ROBDDs to decide whether there exists a valid colouring using three colours.

Reading for next week

CLR Chapter 36.5.