Written Examination, 6th of January 1999 Course no. 49285

Course Name: Advanced Algorithms

Allowed aids: All usual

The problem set consists of 4 problems which are weighted as follows in the total evaluation:
Problem 1: 20%, problem 2: 30%, problem 3: 10%, problem 4: 40%

Marking: 13-scale.

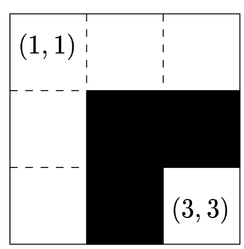
All problems are formulated in both Danish and English.
The Danish version starts on page **??**, the English version starts on page 1.

In this problem set you are asked to give new algorithms. You can use the book's pseudo-code notation or incomplete C/C++/Java-code. Furthermore, you are free to use any data-structures and algorithms from the book and the BDD lecture-notes.

Problem 1 (20%)

In this problem, you are given a *maze* as an $n \times n$ matrix whose entries are either $\sqcup$ or $\blacksquare$. If entry (i, j) is $\sqcup$, position (i, j) of the maze is free, otherwise position (i, j) is blocked. Only horizontal and vertical moves are allowed in the maze (no diagonal moves). Moves can be made only to the positions containing $\sqcup$. You can assume that positions $(1, 1)$ and (n, n) always contain $\sqcup$. A *path* is a sequence of positions and the *length* of a path is the number of positions in the sequence. A *solution* is a path from position $(1, 1)$ to position (n, n). It is possible that no solution exists. For example, the 3×3 matrix to the left corresponds to the maze to the right:

$$\begin{pmatrix} \sqcup & \sqcup & \sqcup \\ \sqcup & \blacksquare & \blacksquare \\ \sqcup & \blacksquare & \sqcup \end{pmatrix}$$



In this example there are no solutions.

The 8×8 matrix to the left corresponds to the maze to the right:

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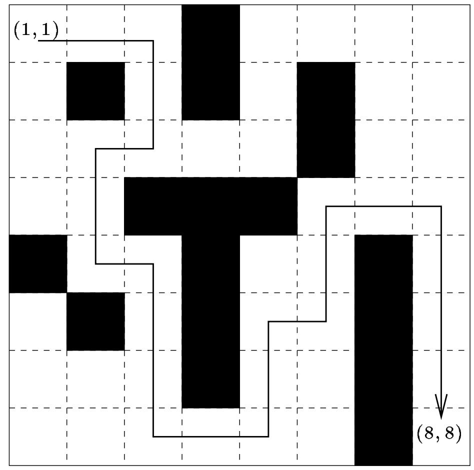
$$\begin{pmatrix}
\sqcup & \sqcup & \sqcup & \blacksquare & \sqcup & \sqcup & \sqcup & \sqcup \\
\sqcup & \blacksquare & \sqcup & \blacksquare & \sqcup & \blacksquare & \sqcup & \sqcup \\
\sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \blacksquare & \sqcup & \sqcup \\
\sqcup & \sqcup & \blacksquare & \blacksquare & \blacksquare & \sqcup & \sqcup & \sqcup \\
\blacksquare & \sqcup & \sqcup & \blacksquare & \sqcup & \sqcup & \blacksquare & \sqcup \\
\sqcup & \blacksquare & \sqcup & \blacksquare & \sqcup & \sqcup & \blacksquare & \sqcup \\
\sqcup & \sqcup & \sqcup & \blacksquare & \sqcup & \sqcup & \blacksquare & \sqcup \\
\sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \blacksquare & \sqcup
\end{pmatrix}$$



In this maze there are many paths from the start position $(1, 1)$ to the finish position $(8, 8)$. A solution of length 25 is shown in the maze to the right.

Question 1.1

Given a maze as an $n \times n$ matrix M, describe how to construct a graph G such that the graph has a path of length $l - 1$ from vertex $(1, 1)$ to vertex (n, n) if and only if there is a solution of length l in the maze.

Solution:

$$\begin{aligned}
V &= \{(i, j) \mid 1 \leq i, j, \leq n, M[i, j] = \sqcup\} \\
E &= \{((i_1, j_1), (i_2, j_2)) \mid (i_1, j_1), (i_2, j_2) \in V \text{ and} \\
&\qquad (|i_1 - i_2| = 1 \ \wedge\ j_1 = j_2) \ \vee\ (i_1 = i_2 \ \wedge\ |j_1 - j_2| = 1)\}
\end{aligned}$$

Question 1.2

Given a maze as an $n \times n$ matrix M, describe an $O(n^2)$ algorithm to find the length of a shortest path from position $(1, 1)$ to position (n, n). Your algorithm should detect if no solution exists.

Solution: Notice, $|V| \leq n^2$ and $|E| \leq 4|V|$. From the matrix, build the graph G, run BFS$(G, (1, 1))$. If (n, n) is white, there is no solution, otherwise $d[n, n] + 1$ is the length of a shortest path. The runtime of BFS is $O(V + E) = O(n^2 + 4n^2) = O(n^2)$.

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