

IT-math F2003 : Supplementary material

Episode 2, February 11, 2003

The Lottery Theorem

Let n and k be natural numbers such that $0 \leq k \leq n$. There are exactly $\binom{n}{k}$ different selections of k distinct numbers from among $1, \dots, n$.

PROOF. We use induction on $n \geq 0$. Here is our induction statement $S(n)$:

For each natural number k such that $0 \leq k \leq n$ there are exactly $\binom{n}{k}$ different selections of k distinct numbers from among $1, \dots, n$.

(looks pretty much like the statement of the theorem).

Let's handle the basis of induction, i.e. prove the statement $S(0)$. Here we assume $n = 0$. Since $0 \leq k \leq n = 0$, we have $k = 0$. There is exactly one way to choose zero numbers from a collection of zero numbers, and $1 = \binom{0}{0} = \binom{n}{k}$, so the basis is OK.

We move on to the induction step. Here one *assumes* $S(n)$ (and calls it the Induction Hypothesis), and uses it to show $S(n+1)$. For the sake of completeness, let's explicitly formulate $S(n+1)$:

For each natural number k such that $0 \leq k \leq n+1$ there are exactly $\binom{n+1}{k}$ different selections of k distinct numbers from among $1, \dots, n+1$.

We split the proof into three cases:

Case 1. $k = 0$.

In this case there is exactly one way to select zero numbers from among $1, \dots, n+1$. Also, $1 = \binom{n+1}{0}$, so we have established $S(n+1)$ in this particular case.

Case 2. $k = n+1$.

In this case there is also exactly one way to select $n+1$ numbers from among $1, \dots, n+1$. Again, $1 = \binom{n+1}{n+1}$, so Case 2 is closed.

Case 3. $0 < k < n+1$.

Let C_1 be the number of selections of k numbers (from among $1, \dots, n+1$) *not* containing the number $n+1$. Let C_2 be the number of those selections of k numbers that *do* contain the number $n+1$.

Then C_1 is equal to the number of ways to select k numbers from among $1, \dots, n$ (since we are not allowed to select $n+1$). By the Induction Hypothesis, $C_1 = \binom{n}{k}$.

The number C_2 is equal to the number of ways to select $k-1$ distinct numbers from among $1, \dots, n$ (since one of the numbers, namely $n+1$ is already pre-selected, and the remaining $k-1$ numbers will have to come from among $1, \dots, n$). By the Induction Hypothesis, $C_2 = \binom{n}{k-1}$.

Clearly, the total number of ways to select k numbers (from among $1, \dots, n+1$) is equal to

$$C_1 + C_2 = \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

(the last equality by the definition of binomial coefficients). So in this case we have also verified $S(n+1)$.

Since our three cases cover all natural numbers k such that $0 \leq k \leq n+1$ (indeed, if $0 \leq k \leq n+1$ then $k = 0$, or $0 < k < n+1$, or $k = n+1$), the induction step is completed and the theorem is established.