

# IT-math F2003 : Supplementary material

## Episode 2, February 11, 2003

### The Lottery Theorem

Let  $n$  and  $k$  be natural numbers such that  $0 \leq k \leq n$ . There are exactly  $\binom{n}{k}$  different selections of  $k$  distinct numbers from among  $1, \dots, n$ .

PROOF. We use induction on  $n \geq 0$ . Here is our induction statement  $S(n)$ :

For each natural number  $k$  such that  $0 \leq k \leq n$  there are exactly  $\binom{n}{k}$  different selections of  $k$  distinct numbers from among  $1, \dots, n$ .

(looks pretty much like the statement of the theorem).

Let's handle the basis of induction, i.e. prove the statement  $S(0)$ . Here we assume  $n = 0$ . Since  $0 \leq k \leq n = 0$ , we have  $k = 0$ . There is exactly one way to choose zero numbers from a collection of zero numbers, and  $1 = \binom{0}{0} = \binom{n}{k}$ , so the basis is OK.

We move on to the induction step. Here one *assumes*  $S(n)$  (and calls it the Induction Hypothesis), and uses it to show  $S(n + 1)$ . For the sake of completeness, let's explicitly formulate  $S(n + 1)$ :

For each natural number  $k$  such that  $0 \leq k \leq n + 1$  there are exactly  $\binom{n + 1}{k}$  different selections of  $k$  distinct numbers from among  $1, \dots, n + 1$ .

We split the proof into three cases:

Case 1.  $k = 0$ .

In this case there is exactly one way to select zero numbers from among  $1, \dots, n + 1$ . Also,  $1 = \binom{n + 1}{0}$ , so we have established  $S(n + 1)$  in this particular case.

Case 2.  $k = n + 1$ .

In this case there is also exactly one way to select  $n + 1$  numbers from among  $1, \dots, n + 1$ . Again,  $1 = \binom{n + 1}{n + 1}$ , so Case 2 is closed.

Case 3.  $0 < k < n + 1$ .

Let  $C_1$  be the number of selections of  $k$  numbers (from among  $1, \dots, n + 1$ ) *not* containing the number  $n + 1$ . Let  $C_2$  be the number of those selections of  $k$  numbers that *do* contain the number  $n + 1$ .

Then  $C_1$  is equal to the number of ways to select  $k$  numbers from among  $1, \dots, n$  (since we are not allowed to select  $n + 1$ ). By the Induction Hypothesis,  $C_1 = \binom{n}{k}$ .

The number  $C_2$  is equal to the number of ways to select  $k - 1$  distinct numbers from among  $1, \dots, n$  (since one of the numbers, namely  $n + 1$  is already pre-selected, and the remaining  $k - 1$  numbers will have to come from among  $1, \dots, n$ ). By the Induction Hypothesis,  $C_2 = \binom{n}{k - 1}$ .

Clearly, the total number of ways to select  $k$  numbers (from among  $1, \dots, n + 1$ ) is equal to

$$C_1 + C_2 = \binom{n}{k} + \binom{n}{k - 1} = \binom{n + 1}{k}$$

(the last equality by the definition of binomial coefficients). So in this case we have also verified  $S(n + 1)$ .

Since our three cases cover all natural numbers  $k$  such that  $0 \leq k \leq n + 1$  (indeed, if  $0 \leq k \leq n + 1$  then  $k = 0$ , or  $0 < k < n + 1$ , or  $k = n + 1$ ), the induction step is completed and the theorem is established.