MATRICES OF RELATIONS

A matrix is a convenient way to represent a relation R from X to Y. Such a representation can be used by a computer to analyze a relation. We label the rows with the elements of X (in some arbitrary order) and we label the columns with the elements of Y (again, in some arbitrary order). We then set the entry in row x and column y to 1 if $x R y$ and to 0 otherwise. This matrix is called the **matrix of the relation R** (relative to the orderings of X and Y).

EXAMPLE 2.6.1 The matrix of the relation

$$R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$$

from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ relative to the orderings 1, 2, 3, 4 and a, b, c, d is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

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EXAMPLE 2.6.2 The matrix of the relation R of Example 2.6.1 relative to the orderings 2, 3, 4, 1 and d, b, a, c is

$$\begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

Obviously, the matrix of a relation from X to Y is dependent on the orderings of X and Y. ■

EXAMPLE 2.6.3 The matrix of the relation R from $\{2, 3, 4\}$ to $\{5, 6, 7, 8\}$, relative to the orderings 2, 3, 4 and 5, 6, 7, 8, defined by

$$x R y \qquad \text{if } x \text{ divides } y$$

is

$$\begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

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When we write the matrix of a relation R on a set X (i.e., from X to X), we use the same ordering for the rows as we do for the columns.

EXAMPLE 2.6.4 The matrix of the relation

$$R = \{(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\}$$

on $\{a, b, c, d\}$, relative to the ordering a, b, c, d, is

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

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Notice that the matrix of a relation on a set X is always a square matrix.

We can quickly determine whether a relation R on a set X is reflexive by examining the matrix A of R (relative to some ordering). The relation R is reflexive if and only if A has 1's on the main diagonal. (The main diagonal of a square matrix consists of the entries on a line from the upper left to the lower right.) The relation R is reflexive if and only if $(x, x) \in R$ for all $x \in X$. But this last condition holds precisely when the main diagonal consists of 1's. Notice that the relation R of Example 2.6.4 is reflexive and that the main diagonal of the matrix of R consists of 1's.

We can also quickly determine whether a relation R on a set X is symmetric by examining the matrix A of R (relative to some ordering). The relation R is symmetric if and only if for all i and j, the ijth entry of A is equal to the jith entry of A. (Less formally, R is symmetric if and only if A is symmetric about the main diagonal.) The reason is that R is symmetric if and only if whenever (x, y) is in R, (y, x) is also in R. But this last condition holds precisely when A is symmetric about the main diagonal. Notice that the relation R of Example 2.6.4 is symmetric and that the matrix of R is symmetric about the main diagonal.

We can also quickly determine whether a relation R is antisymmetric by examining the matrix of R (relative to some ordering).