

IT-math F2003 : Test Exam

The duration of the exam is 4 hours.

You are allowed to use your notes and any literature, and/or calculator.

Remember to write name of course, your name, personal ID, page number, and total number of pages, as in 'IT-math, J.Bond, ID 007, page 12 of 36' on each page you hand in.

All problems including the extra one carry equal weight; the extra problem, however, is expected to require a tad more creativity than others.

1. Prove that for each natural number $n \geq 1$ we have

$$1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2.$$

2. Let A and B be finite sets. Show that $|\mathcal{P}(A) \cap \mathcal{P}(B)| = 2^{|A \cap B|}$.

[Recall that $\mathcal{P}(X)$ is the set of all subsets of X .]

3. Let $R \subseteq X \times X$ be a relation on a non-empty set X . Suppose that R is a partial ordering as well as an equivalence relation on X . Show that R is then also a function from X to X .
4. Find integers x and y such that $74x + 118y = 2$.
5. Show that $n^2 + n \cdot (\log_2 n)^{42} = \Theta(n^2)$.
6. Construct a context-free grammar G such that

$$\mathcal{L}(G) = \{a^n b^n c^k \mid n, k \in \mathbb{N}\}.$$

[Remember that $0 \in \mathbb{N}$.]

7. Let \mathcal{R} be the set of *all* regular languages over the alphabet $\{a, b\}$. Consider the function $f : \mathcal{R} \rightarrow \mathcal{R}$ defined by $f(L) = \{a\}^* \circ L$. Is the function f injective? Motivate your answer. [$\{a\}$ is the language consisting of the single one-letter word a . $*$ is the Kleene star. \circ denotes concatenation of languages.]

Extra Problem

8. Given 5 distinct points in the plane, each with integer coordinates, consider the intervals connecting each pair of these points. Show that the midpoint of at least one of these intervals has integer coordinates.

[Recall that the midpoint of the interval connecting two points with coordinates (x_1, y_1) and (x_2, y_2) has coordinates $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.]