IT-math F2003 : Classroom Exercises
Episode 11, April 22, 2003

1. Let a and b be real numbers such that $a > b > 1$. Show that $a^n = o(b^n)$.

2. Find an error in the following argument that purports to establish $2^n = o\left((\frac{3}{2})^n\right)$:
   By the Binomial Theorem, we have
   $$2^n = (1+1)^n = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + 1.$$
   By the formula $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$, each term in the sum on the right hand side of the above equality is a function that is a polynomial in n, and hence $o\left((\frac{3}{2})^n\right)$. But the sum of several functions that are $o\left((\frac{3}{2})^n\right)$ must itself be $o\left((\frac{3}{2})^n\right)$. Hence $2^n = o\left((\frac{3}{2})^n\right)$.

3. Let $k \in \mathbb{N}_+$. Show that $\sum_{1 \le i \le n} i^k = \Theta(n^{k+1})$.

4. Construct automata that accept the following languages in the alphabet $A = \{a, b\}$:
   (a) $\varnothing \subseteq A^*$, the empty language, i.e. the language containing no words whatsoever;
   (b) $\{\, w \in A^* \mid w$ has an even number of occurrences of $a \,\}$;
   (c) $\{aab\}$, the language containing the single word aab.

IT-math F2003 : Homework Exercises
Episode 11, April 22, 2003

Fisherperson's Exercises

1. For each of the following, select a Theta notation from among $\Theta(n)$, $\Theta(n \log n)$, $\Theta(n^2)$, $\Theta(n^3/\log n)$, and motivate your answer:
   (a) $2\log n + 4n + 3n\log n$;
   (b) $2 + 4 + 6 + \cdots + 2n$;
   (c) $\frac{(n+1)(n+3)}{n+2}$;
   (d) $\frac{(n^2+\log n)(n+1)}{n+n^2}$.

2. Suppose $f = O(g)$ and let h be any function. Prove that $f \cdot h = O(g \cdot h)$. Similarly for Ω and Θ in place of O.

   [The function $f \cdot h$ is defined by $(f \cdot h)(n) = f(n) \cdot h(n)$.]

3. Draw (the transition diagram of) the automaton given by $A = \{a, b\}$, $S = \{\sigma_0, \sigma_1, \sigma_2\}$, $I = \sigma_0$, $T = \{\sigma_0\}$, $f(\sigma_i, a) = \sigma_{i+1 \bmod 3}$, $f(\sigma_i, b) = \sigma_{i\cdot(i-1) \bmod 3}$. Does this automaton accept the word *abbabb*?

Snake-Charmer's Exercises

1. Prove or refute:
   (a) $\log_2 n = o(3^n)$;
   (b) $n^4(\log_2 n)^{12} = \Omega(n^5 \log_2 n^6)$;
   (c) $3^n - n - 1000(\log n)^2 = \Omega(2^n)$.

2. Suppose $f = o(g)$ and that the function h is such that $h(n) \neq 0$ for almost all n. Prove that $f \cdot h = o(g \cdot h)$.

3. For any real $a > 1$ show that $a^n = o(n!)$.

4. Construct an automaton (over the alphabet $\{a, b\}$) accepting the language $\{\varepsilon\}$, i.e. the language having the empty word as its only element.

Lion-Hunter's Exercises

1. Show that $2^n = o\left(\binom{2n}{n}\right)$.

2. Prove or refute: $\binom{2n}{n} = O(n^{\log_2 n})$.

3. Construct an automaton accepting the language

   $$\{\, w \in \{a, b\}^* \mid w \text{ has an odd number of occurrences of the subword } ab \,\}.$$

   [The number of occurences of the subword *ab* is the number of those occurrences of the letter *a* that are *immediately* followed by an occurrence of the letter *b*.]

Dragonslayer's Exercise

1. Construct an automaton accepting the language

   $$\{\, w \in \{0, \ldots, 9\}^* \mid w \text{ is a decimal representation of a natural number divisible by } 3 \,\}.$$