IT-math F2003 : Classroom Exercises
Episode 7, March 18, 2003

1. Draw the Hasse diagrams of
   (a) (Div(45), |);
   (b) ($\mathcal{P}(\{1\})$, $\subseteq$).

2. Determine whether the following are equivalence relations on $X = \{1, 2, 3, 4, 5\}$. If the relation is an equivalence relation, write down its equivalence classes, and the corresponding partition of X:
   (a) $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)\}$;
   (b) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

3. Is $\{\{1, 3, 5\}, \{2, 4\}\}$ a partition of $\{1, \ldots, 5\}$? If it is, write down the equivalence classes of the corresponding equivalence relation.

4. Let the relation R on $\mathbb{Z} \times \mathbb{Z}$ be defined by

$$(m, n)\ R\ (m', n') \quad \Longleftrightarrow \quad m \cdot n' = m' \cdot n.$$

   Is R an equivalence relation?

IT-math F2003 : Homework Exercises
Episode 7, March 18, 2003

Fisherperson's Exercises

1. Draw the Hasse diagrams of
   (a) (Div(180), $|$);
   (b) ($\mathcal{P}(\{0,1,2,3\})$, $\subseteq$).

2. Determine if
   (a) $\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1),(3,4),(4,3)\}$;
   (b) $\{(1,1),(1,4),(4,1),(2,2),(2,5),(5,2),(3,3),(4,4),(5,5)\}$
   are equivalence relations on $X = \{1,2,3,4,5\}$. Draw the matrices of both relations. For equivalence relations, write down the corresponding partition(s).

3. Determine if $\{\,\{n\} \mid n \in \mathbb{N}\,\}$ is a partition of $\mathbb{N}$.

Snake-Charmer's Exercises

1. Given an equivalence relaton E on a set X, recall the set
   $$X/E = \{\,[x]_E \mid x \in X\,\}, \qquad \text{where} \quad [x]_E = \{\,y \in X \mid y\ E\ x\,\}.$$
   Write down a careful proof that X/E is a partition of X.

2. Suppose the relation R on a set X is a partial ordering as well as an equivalence relation. Show that for $x, y \in X$, one has $x\ R\ y$ iff $x = y$.

3. Suppose E is an equivalence relation on a set X with $|X| = 12$, and E has two (distinct) equivalence classes each containing 5 elements, and an equivalence class with two elements. Calculate $|E|$, the total number of pairs in E.

Lion-Hunter's Exercises

1. Let the relation R on $\mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ be defined by
   $$(m,n)\ R\ (m',n') \quad \Longleftrightarrow \quad m \cdot n' = m' \cdot n.$$
   Is R an equivalence relation?

2. Let X be a set and $Y \subseteq X$. Define a relation R on $\mathcal{P}(X)$ by
   $$A\ R\ B \quad \Longleftrightarrow \quad A \bigtriangleup B \subseteq Y.$$
   (Recall that $\bigtriangleup$ is the symmetric difference: $A \bigtriangleup B = (A \cup B) \setminus (A \cap B)$.) Show that R is an equivalence relation on $\mathcal{P}(X)$.

3. Suppose $|X| = 4$. Consider the set $\mathcal{E}$ of all equivalence realtions on X. This set is partially ordered by $\subseteq$. Draw the Hasse diagram of the poset ($\mathcal{E}$, $\subseteq$).

Dragonslayer's Exercise

1. For $n, k \in \mathbb{N}_+$ with $k \leq n$, let $\left\{ {n \atop k} \right\}$ be the number of equivalence relations on a set with n elements having exactly k equivalence classes (these are known as *Stirling numbers of the second kind*).
   (a) Show that $\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1$;
   (b) For $1 \leq k < n$, show that $\left\{ {n+1 \atop k+1} \right\} = \left\{ {n \atop k} \right\} + (k+1) \cdot \left\{ {n \atop k+1} \right\}$;
   (c) Compute $\left\{ {5 \atop 3} \right\}$;
   (d) For $n \geq 2$, show that $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$;
   (e) For $n \geq 2$, show that $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$.