IT-math F2003 : Classroom Exercises
Episode 8, March 25, 2003

1. Determine whether each of the following is a function from $X = \{1, 2, 3\}$ to $\{a, b, c\}$ (Assume that a, b, c are distinct).
   (a) $\{(1, a), (1, b), (2, b), (3, c)\}$;
   (b) $\{(1, a), (1, a), (2, b), (3, c)\}$;
   (c) $\{(1, b), (2, c)\}$.

2. Consider the function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ defined by putting $f([x]_5) = [4]_5 \cdot [x]_5$.
   (a) Draw the digraph of f;
   (b) Is f one-to-one and/or onto?

3. If $f : X \to Y$ is a function, show that
   (a) $E = \{ (x_1, x_2) \in X^2 \mid f(x_1) = f(x_2) \}$ is an equivalence relation on X;
   (b) $X/E = \{ f^{-1}[y] \mid y \in \operatorname{rng} f \}$.
   (Recall that $\operatorname{rng} f = \{ y \in Y \mid f(x) = y \text{ for some } x \in X \}$.)

4. If 7 colours are used to paint 50 bikes, prove that there are at least 8 bikes that are painted the same colour.

5. Suppose $X \subseteq \{1, \ldots, 115\}$ and $|X| = 60$. Show that there are $y, z \in X$ with $y - z = 4$.

   [Hint: Use the Pigeonhole Principle.]

6. Let E be an equivalence relation on X. Consider the function $\nu : X \to X/E$ given by $\nu(x) = [x]_E$. Show that ν is onto.

IT-math F2003 : Homework Exercises
Episode 8, March 25, 2003

Fisherperson's Exercises

1. Find out if the following relations from X to Y are functions from X to Y. For function(s) between finite sets, draw the arrow diagram(s).
   (a) $\left\{(x, y) \in \mathbb{Z}^2 \,\middle|\, x, y \in \{0, \ldots, 5\} \text{ and } x + y = 5\right\}$, where $X = Y = \{0, \ldots, 5\}$;
   (b) $\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 = y^2\right\}$, where $X = Y = \mathbb{R}$;
   (c) $\left\{([x]_6, [y]_2) \in \mathbb{Z}_6 \times \mathbb{Z}_2 \,\middle|\, [x]_6 \subseteq [y]_2\right\}$, where $X = \mathbb{Z}_6$, $Y = \mathbb{Z}_2$.

2. Determine whether the following functions are one-to-one and/or onto:
   (a) $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 6x - 9$;
   (b) $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$;
   (c) $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n + 2$;
   (d) $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = n + 2$;
   (e) $f : \mathcal{P}(\{0, 1\}) \to \mathcal{P}(\{0, 1\})$ defined as $f(X) = X \triangle \{1\}$.

3. Show that among any 30 people there are at least 5 who were born on the same day of the week.

Snake-Charmer's Exercises

1. Show that every set of 15 socks chosen from among 13 pairs of socks contains at least one matching pair.

2. Show that among any six integers in $\{1, \ldots, 10\}$ you can find two whose sum is equal to 11.

3. Give an example of an onto function from $\mathbb{R}$ to $\mathbb{Z}$.

Lion-Hunter's Exercises

1. Let $a_1, \ldots, a_n \in \mathbb{Z}$. Show that there are i, j with $1 \leq i \leq j \leq n$ such that $\sum_{k=i}^{j} a_k$ is divisible by n.

2. Let E be an equivalence relation on X. Suppose the function $\mu : X/E \to X$ is such that $\mu(C) \in C$ for all $C \in X/E$. Show that μ is one-to-one.

3. Let $m, n \in \mathbb{N}_+$, and consider the function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $f([x]_n) = [m]_n \cdot [x]_n$. Show that f is one-to-one if and only if f is onto if and only if $\gcd(m, n) = 1$.

Dragonslayer's Exercise

1. Let X be an arbitrary 8-element subset of $\{1, \ldots, 20\}$. Show that there are two distinct 3-element subsets S and T of X such that the sum of elements of S is equal to that of elements of T.