# IT-math F2003 : Classroom Exercises
# Episode 9, April 1, 2003

1. Let $f$ and $g$ be functions from $X$ to $Y$. Show that $f = g$ if and only if for all $x \in X$ one has $f(x) = g(x)$.

2. Suppose $Y$ is a set, and that $X = \{x_1, \ldots, x_k\}$ is a set with exactly $k$ elements. Define a function $F : Y^X \to Y^k$ by $F(f) = (f(x_1), \ldots, f(x_k))$. Show that $F$ is bijective.

   [Recall that $X^k = X \times \cdots \times X$, and that $Y^X$ is the set of all functions from $X$ to $Y$.]

3. Let $A$ be a set and suppose the function $F : \mathcal{P}(A) \to \{0,1\}^A$ is defined by

$$F(X)(a) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{if } a \notin X \end{cases}.$$

   Show that $F$ is a bijection.

   [Recall that $\mathcal{P}(A)$ is the set of all subsets of $A$.]

4. Suppose the functions $f, g : \mathbb{N} \to \mathbb{N}$ are given by the formulas $f(n) = n^2$ and $g(n) = 2^n$. Find the formulas for the following:
   (a) $f \circ f$;
   (b) $f \circ g$;
   (c) $g \circ f$;
   (d) $g \circ g$.

# IT-math F2003 : Homework Exercises
# Episode 9, April 1, 2003

## Fisherperson's Exercises

1. Which of the following functions are bijections?
   (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$;
   (b) $c : \mathcal{P}(A) \to \mathcal{P}(A)$, where $A$ is a set and $c(X) = A \setminus X$;
   (c) $g : \mathbb{N} \to \mathbb{N}$ defined by $g(n) = \binom{n+2}{2}$;
   (d) $q : \mathbb{Z}_9 \to \mathbb{Z}_9$ defined by $q([x]_9) = [4x]_9$;
   (e) $i : \mathbb{N} \to \mathbb{Z}$ defined by $i(n) = n$.

2. Let us look at four-letter words that only use letters A, B, C, D, E, F.
   (a) How many four-letter words are there without repetition of letters?
   (b) How many four-letter words are there if we allow repetition of letters?
   (c) How many four-letter words begin with A and do not have repetitions?
   (d) How many four-letter words use the letter A at least once?
   Motivate your answers.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is *linear* if there are $a, b \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ one has $f(x) = ax + b$. Show that the composition of two linear functions is a linear function.

## Snake-Charmer's Exercises

1. If $f : X \to Y$ and $g : Y \to Z$ are bijections, show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

2. Given the function $f = \{(a, b), (b, a), (c, b)\}$ from $X = \{a, b, c\}$ to $X$ ($a, b, c$ all distinct), write $f \circ f$ and $f \circ f \circ f$ as sets of pairs. Further, define $f^{(1)} = f$, $f^{(n+1)} = f \circ f^{(n)}$, and write $f^{(678)}$ as a set of pairs.

3. Construct a bijection between $\mathbb{N}$ and $\{n \in \mathbb{N} \mid n \text{ odd}\}$.

   [Hint: Use a linear function.]

## Lion-Hunter's Exercises

1. Let $X$ and $Y$ be non-empty subsets of $\mathbb{N}$. Prove that there exists a 1–1 function from $X$ to $Y$ if and only if there exists an onto function from $Y$ to $X$.

2. Let $|X| = n$. Consider the set

$$\mathcal{Z} = \{ (A, B) \mid A \subseteq B \subseteq X \},$$

   i.e. the set of all pairs $(A, B)$ of subsets of $X$ with such that the first element of the pair is a subset of the second one. Show that $|\mathcal{Z}| = 3^n$.

   [Hint: Construct a bijection between $\mathcal{Z}$ and $\{0, 1, 2\}^X$.]

3. Prove that a function $f : X \to Y$ is onto if and only if for any set $Z$ and functions $g, h : Y \to Z$ one has that $g \circ f = h \circ f$ implies $g = h$.

## Dragonslayer's Exercise

1. Let $k, n \in \mathbb{N}_+$ be such that $k \leq n$. Suppose $|X| = n$ and $|Y| = k$. Show that there are $k! \cdot \left\{ {n \atop k} \right\}$ many surjections from $X$ to $Y$.

   [$\left\{ {n \atop k} \right\}$ are Stirling numbers of the 2nd kind—see DS1 in Episode 7.]