IT-math F2003 : Classroom Exercises
Episode 2, February 11, 2003

1. Write down a recursive (inductive) definition of exponentiation x^n for natural n.
2. What is the largest possible number of edges in a simple graph with n vertices?
3. Prove that
$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$
(Suggestion: Induction on $n \geq k$.)
4. Write down the sum $1 + 2 + \cdots + n$ with the help of $\sum$-notation.
5. Write down the Binomial Theorem with the use of the $\sum$-notation.
6. Write out $\displaystyle\sum_{k=0}^{3} k \cdot 2^k$ without the use of the $\sum$-notation and calculate it.
7. Write down a recursive (inductive) definition of $\sum_{i=k_1}^{k_2} f(i)$.
8. What is the largest possible number of regions that n straight lines can divide a plane into?

IT-math F2003 : Homework Exercises
Episode 2, February 11, 2003

to be submitted on or before February 18, 2003

Fisherperson's Exercises

1. In how many ways can a six-card hand be dealt from a deck of 52 cards? (You don't need to calculate the precise number: an expression for the number will suffice.)

2. Calculate 11^7 using the Binomial Theorem (Hint: $11 = 10 + 1$).

3. The notation $\displaystyle\prod_{k=a}^{b} f(k)$ works exactly like the $\sum$-notation, but instead of the sum you take the product: $\displaystyle\prod_{k=a}^{b} f(k) = f(a) \cdot f(a+1) \cdot \cdots \cdot f(b)$.
   (a) Find $\prod_{n=1}^{3}(n^2 - 3n + 3)$;
   (b) Find $\prod_{n=0}^{2} 2^2$;
   (c) Write down $n!$ using the $\prod$ notation, but without the use of either $\cdots$ or !.

Snake-Charmer's Exercises

1. Use the Binomial theorem to show that for integers $n > 0$ we have $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

2. Prove by induction that $1 + 5 + 9 + \cdots + (4k - 3) = k \cdot (2k - 1)$ for all integers $k \geq 1$.

3. Show that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ for all natural numbers k.

4. The *Fibonacci numbers* f_n are defined for natural numbers n by the following recursive definition:
$$f_n = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 1; \\ f_{n-2} + f_{n-1} & \text{otherwise.} \end{cases}$$
   Show by induction that $\sum_{k=0}^{n} f_k = f_{n+2} - 1$.

Lion-Hunter's Exercises

1. Show that $n! \neq 2^n$ for all integers $n \geq 4$.

2. Show that by using only 2- and 7-øre stamps one can pay the postage of n øre for any integer $n \geq 6$.

3. If you look at the rows of Pascal's triangle you'll notice that the numbers in a row first strictly increase, then reach a maximum that is achieved by one or two numbers in the centre of the row, and then strictly decrease. Try to formulate this observation as a precise mathematical statement/conjecture.

Dragonslayer's Exercise

1. Prove for all natural numbers n that
$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}.$$