IT-math F2003 : Selected Solution(s)
Episode 1, February 4, 2003

LH1. Prove by induction on $n \geq 0$ that $\binom{n}{k} = \binom{n}{n-k}$ for all integers k such that $0 \leq k \leq n$.

Solution. First, the induction basis, $n = 0$: Since $0 \leq k \leq n$, there is only $k = 0$ to consider. We have : $\binom{n}{k} = \binom{0}{0} = 1$ by the inductive definition of binomial coefficients, and $\binom{n}{n-k} = \binom{0}{0} = 1$, so $\binom{n}{k} = \binom{n}{n-k}$ in this case.

For the induction step, assume $\binom{n}{k} = \binom{n}{n-k}$ for all integers k such that $0 \leq k \leq n$. We've got to show: $\binom{n+1}{k} = \binom{n+1}{n+1-k}$ for all integers k satisfying $0 \leq k \leq n+1$. We split the proof into three cases:

Case 1. $k = 0$.
We have $\binom{n+1}{k} = \binom{n+1}{0} = 1$ by definition, and also $\binom{n+1}{n+1-k} = \binom{n+1}{n+1} = 1$, so $\binom{n+1}{k} = \binom{n+1}{n+1-k}$.

Case 2. $k = n+1$.
Similar to Case 1: $\binom{n+1}{k} = \binom{n+1}{n+1} = 1$, and also $\binom{n+1}{n+1-k} = \binom{n+1}{0} = 1$.

Case 3. $0 < k < n+1$.
We have:

$$\binom{n+1}{k} \overset{(*)}{=} \binom{n}{k-1} + \binom{n}{k} \overset{\text{(I.H.)}}{=} \binom{n}{n-(k-1)} + \binom{n}{n-k} = \binom{n}{n-k} + \binom{n}{(n-k)+1}$$

$$\overset{(*)}{=} \binom{n+1}{(n-k)+1} = \binom{n+1}{(n+1)-k},$$

where equalities marked by $(*)$ follow by the inductive definition of the binomial coefficients, and (I.H.) marks an application of the induction hypothesis.

Thus we get the desired result in all three cases, and so the induction step, and hence the solution, are completed.

DS1. Show by induction on $n \geq 1$ that $3^n + 7^n - 2$ is divisible by 8.

Solution. The basis for $n = 1$ is easy: $3^1 + 7^1 - 2 = 3 + 7 - 2 = 8$, which is clearly divisible by 8.

For the induction step, we assume that $3^n + 7^n - 2$ is divisible by 8 and we are going to show that $3^{n+1} + 7^{n+1} - 2$ also is. As

$$3^{n+1} + 7^{n+1} - 2 = \left((3^{n+1} + 7^{n+1} - 2) - (3^n + 7^n - 2)\right) + (3^n + 7^n - 2),$$

it will suffice to show that $(3^{n+1} + 7^{n+1} - 2) - (3^n + 7^n - 2)$ is divisible by 8, for we know that $3^n + 7^n - 2$ is so divisible by the induction hypothesis, and so $3^{n+1} + 7^{n+1} - 2$ will have to be divisible by 8 as the sum of two numbers both divisible by 8. We have:

$$(3^{n+1} + 7^{n+1} - 2) - (3^n + 7^n - 2) = 3^{n+1} - 3^n + 7^{n+1} - 7^n = (3-1) \cdot 3^n + (7-1) \cdot 7^n = 2 \cdot 3^n + 6 \cdot 7^n$$

$$= 4(3^n + 7^n) + 2(7^n - 3^n) = 4(3^n + 7^n) + 2(7-3)(7^{n-1} + 7^{n-2} \cdot 3 + \cdots + 7 \cdot 3^{n-2} + 3^{n-1}).$$

(The last step here uses the identity $a^n - b^n = (a-b) \cdot (a^{n-1} + a^{n-2} \cdot b + \cdots + a \cdot b^{n-2} + b^{n-1})$.) Now, both 3^n and 7^n are odd as positive integer powers of odd numbers, so their sum is even. Sinnce 4 times an even number is divisible by 8, the term $4(3^n + 7^n)$ is divisible by 8. The term $2(7-3)(7^{n-1} + 7^{n-2} \cdot 3 + \cdots + 7 \cdot 3^{n-2} + 3^{n-1})$ is divisible by 8 as it has the form $2(7-3) = 8$ times some integer. Therefore $(3^{n+1} + 7^{n+1} - 2) - (3^n + 7^n - 2)$, being the sum of two numbers divisible by 8, is itself divisible by 8. By the discussion at the beginning of the induction step, this implies that $3^{n+1} + 7^{n+1} - 2$ is divisible by 8.