IT-math F2003 : Selected Solution(s)
Episode 2, February 11, 2003

SC1. Use the Binomial theorem to show that for integers $n > 0$ we have $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

Solution. The Binomial Theorem tells us that

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}.$$

Put $x = -1$ and $y = 1$. For integers $n > 0$ you get

$$0 = 0^n = (-1+1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}$$

because $1^n = 1$ for any natural number n.

SC3. Show that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ for all natural numbers k.

Solution. Basis: $\sum_{k=0}^{0} 2^k = 2^0 = 1 = 2^{0+1} - 1$.

Induction step: Assume $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$.

$$\sum_{k=0}^{n+1} 2^k = \left(\sum_{k=0}^{n} 2^k\right) + 2^{n+1} \overset{\text{(I.H.)}}{=} 2^{n+1} - 1 + 2^{n+1} = 2\cdot 2^{n+1} - 1 = 2^{(n+1)+1} - 1.$$

LH2. Show that by using only 2- and 7-øre stamps one can pay the postage of n øre for any integer $n \geq 6$.

Solution. Basis: 6 øre is paid by three 2-øre stamps.

Induction step: Suppose a collection S of 2- and 7-øre stamps pays the postage of $n \geq 6$ øre. How does one get a new collection that pays $n + 1$ øre?

If there is a 7-øre stamp in S, replace it by four 2-øre stamps. The worth of the new collection is $n - 7 + 4\cdot 2 = n + 1$ øre.

If there is no 7-øre stamp in S, then there must be at least three 2-øre stamps, as the postage paid by S is $n \geq 6$ øre. So, replace three 2-øre stamps by a single 7-øre stamp. The new collection pays $n - 3\cdot 2 + 7 = n + 1$ øre.

The induction step, and hence the solution, are completed.

LH3. If you look at the rows of Pascal's triangle you'll notice that the numbers in a row first strictly increase, then reach a maximum that is achieved by one or two numbers in the centre of the row, and then strictly decrease. Try to formulate this observation as a precise mathematical statement/conjecture.

Solution. For all natural numbers n, j and k such that $0 \leq j < k \leq \frac{n}{2}$ there holds $\binom{n}{j} < \binom{n}{k}$.

(We can add: For all natural numbers n, j and k such that $\frac{n}{2} \leq j < k \leq n$ there holds $\binom{n}{k} < \binom{n}{j}$, but this is really equivalent to the above conjecture by Homework Exercise LH1 from Episode 1.)