IT-math F2003 : Selected Solution(s)
Episode 3, February 18, 2003

FP2(b). Find out whether $\{2,3\} \subseteq \{\{2,2,3\}\}$ (and motivate your answer):

Solution. This is not true, as the set $\{2,3\}$ has tw odistinct elements 2 and 3, while the set $\{\{2,2,3\}\}$ has just one element, namely $\{2,2,3\}$, so $\{2,3\}$ cannot be a subset of $\{\{2,2,3\}\}$.

SC1. Prove or refute: For any sets A, B, C we have $(A \setminus B) \setminus C = A \setminus (B \setminus C)$.

Solution. **(T. Skjødt)** We shall refute this by giving an individual example of sets A, B, and C such that $(A \setminus B) \setminus C \neq A \setminus (B \setminus C)$.

The example is: $A = \{1,2,3\}$, $B = \{3,4,5\}$, $C = \{1,5,6\}$. Then

$$(A \setminus B) \setminus C = \{1,2\} \setminus \{1,5,6\} = \{1\} \neq \{1,2\} = A = \{1,2,3\} \setminus \{3,4\} = A \setminus (B \setminus C).$$

SC2. Prove that, for any sets A, B with $A, B \subseteq X$ and the complement $(\cdot)^c$ being relative to X, the following statements are equivalent (that is, any two of the following statements can be connected by an 'iff'):

(i) $A \subseteq B$;
(ii) $A \cap B = A$;
(iii) $A \cup B = B$;
(iv) $B^c \subseteq A^c$.

Solution. We shall show the following equivalences: (i)$\Longleftrightarrow$(ii), (i)$\Longleftrightarrow$(iii), and (i)$\Longleftrightarrow$(iv), so that the remaining ones follow by combining an appropriate pair of the ones we prove. [Comments in brackets in the paragraphs below are there hoping that they may help some students better understand the mechanics of these proofs. Students don't have to include comments like these in written homework or solutions of exam problems.]

(i)$\Longrightarrow$(ii): Suppose $A \subseteq B$. [To prove $A = A \cap B$, we have to show two things: $A \subseteq A \cap B$ and $A \cap B \subseteq A$]. Let us show that $A \subseteq A \cap B$. [To show $A \subseteq A \cap B$, one takes an arbitrary $x \in A$ and proves that $x \in A \cap B$.] For any $x \in A$ we also have $x \in B$, for $A \subseteq B$. So we get $x \in A$ and $x \in B$. Thus $x \in A \cap B$. Therefore $A \subseteq A \cap B$. As $A \cap B \subseteq A$ holds for any two sets A and B [we've seen as much in the lecture: if $x \in A \cap B$ then $x \in A$ and $x \in B$, so in particular $x \in B$.], we conclude $A \cap B = A$.

(ii)$\Longrightarrow$(i): Suppose we have $A \cap B = A$. [To show $A \subseteq B$, one takes an arbitrary $x \in A$ and shows that this x is an element of B as well.] Let $x \in A$. Then $x \in A \cap B$ since $A \cap B = A$. In particular, $x \in B$ [$x \in A \cap B$ means that $x \in A$ and $x \in B$, so we do have $x \in B$.]. Thus we've shown that $A \subseteq B$.

(i)$\Longrightarrow$(iii): Suppose $A \subseteq B$. Let us show that $A \cup B \subseteq B$. [Again, take an arbitrary element of $A \cup B$ and show that it is in B.] If $x \in A \cup B$, then $x \in A$ or $x \in B$. If $x \in B$, we are done. If $x \in A$ then it follows that $x \in B$ again because $A \subseteq B$. Thus we get $x \in B$ in either case. This shows $A \cup B \subseteq B$.

(iii)$\Longrightarrow$(i): Suppose we have $A \cup B = B$. [We are now going to show that $A \subseteq B$ by the usual method.] Let $x \in A$. Then $x \in A \cup B$ for one has $A \subseteq A \cup B$ for any A, B [seen in the lecture; if $x \in A$, then $x \in A$ or $x \in B$]. But this implies $x \in B$ as $B = A \cup B$. Thus we have shown $A \subseteq B$.

(i)$\Longrightarrow$(iv): Suppose $A \subseteq B$. [To show $B^c \subseteq A^c$ we take an arbitrary $x \in B^c$ and show $x \in A^c$.] Let $x \in B^c$. Recall that $B^c = X \setminus A$. Reason by contradiction: If $x \in A$ then since $A \subseteq B$ we also have $x \in B$. But then $x \notin B^c$ which contradicts the assumption and hence implies that $x \in A$ cannot possibly be the case. So $x \notin A$, which, since $x \in X$, means $x \in X \setminus A = A^c$, so $x \in A^c$. Thus $B^c \subseteq A^c$

(iv)$\Longrightarrow$(i): Assume $B^c \subseteq A^c$. Since we have already proved (i)$\Longrightarrow$(iv), we may apply this to B^c in the role of [(i)$\Longrightarrow$(iv)'s] A and A^c in the role of [(i)$\Longrightarrow$(iv)'s] B. We get $(A^c)^c \subseteq (B^c)^c$. Recalling [from the lecture] that $(A^c)^c = A$ and $(B^c)^c = B$, we conclude $A \subseteq B$ as required.

LH3. For all sets A, B, C, prove that if $A \times B = A \times C$, then $A = \emptyset$ or $B = C$.

Solution. Suppose $A \times B = A \times C$. If $A = \emptyset$, there is nothing to prove, so assume $A \neq \emptyset$. We shall show $B = C$.

Since $A \neq \emptyset$, we can fix an element $a \in A$. Let $b \in B$. Then $(a, b) \in A \times B$. Since $A \times B = A \times C$, we also have $(a, b) \in A \times C$. Therefore $b \in C$. We have shown $B \subseteq C$. In an entirely similar fashion, one shows $C \subseteq B$. Therefore $B = C$. The proof is complete.

DS1. Refute: There is a set X such that $X = \mathcal{P}(X)$.

Solution. We are going to refute this by showing that $X = \mathcal{P}(X)$ leads to a contradiction. So suppose $X = \mathcal{P}(X)$. Consider $Z = \{ Y \in \mathcal{P}(X) \mid Y \notin Y \}$. We have $Z \subseteq \mathcal{P}(X) = X$, so $Z \subseteq X$, therefore $Z \in \mathcal{P}(X)$. If $Z \in Z$ then, by the definition of Z, one has $Z \notin Z$. If $Z \notin Z$ then $Z \in Z$ again, by the definition of Z. Thus we have $Z \in Z$ if and only if $Z \in Z$. This is the desired contradiction.