

IT-math F2003 : Selected Solution(s)

Episode 6, March 11, 2003

SC1. Let $b > 1$ be an integer, $K \in \mathbb{N}$, and let n_i be some base- b digits: $0 \leq n_i < b$. Show that $\sum_{i=0}^K n_i \cdot b^i < b^{K+1}$.

Solution. We use induction on $K \geq 0$.

For $K = 0$ we have $\sum_{i=0}^0 n_i \cdot b^i = n_0 \cdot b^0 = n_0 < b = b^{0+1}$ as required.

Assuming the inequality holds for K , let us show that it also holds for $K + 1$. We do this by calculating

$$\sum_{i=0}^{K+1} n_i \cdot b^i = \left(\sum_{i=0}^K n_i \cdot b^i \right) + n_{K+1} \cdot b^{K+1} \stackrel{(\text{I.H.})}{<} b^{K+1} + n_{K+1} \cdot b^{K+1} \leq (n_{K+1} + 1) \cdot b^{K+1} \leq b \cdot b^{K+1} = b^{(K+1)+1}.$$

SC2. Give an example of a relation on \mathbb{N} that is both symmetric and antisymmetric.

Solution. The relation I (a.k.a. $=$) on \mathbb{N} defined by $n I m \iff n = m$ is symmetric because $n = m$ always implies $m = n$. The relation I is also antisymmetric, for if we have $n = m$ and $m = n$, then clearly $n = m$.

SC3. (C. Pedersen) Recall the relation Q on \mathbb{R}^2 defined by

$$(x, y) Q (z, w) \iff x \leq z \text{ and } y \leq w.$$

Show that Q is transitive.

Solution. (T. Majeed) To show that Q is transitive, let $(x, y), (u, v), (z, w) \in \mathbb{R}^2$, and assume that $(x, y) Q (u, v)$ and $(u, v) Q (z, w)$. [We have to show that this implies $(x, y) Q (z, w)$.] By the definition of Q , this means that $x \leq u$ and $y \leq v$ [this follows from $(x, y) Q (u, v)$], as well as $u \leq z$ and $v \leq w$ [this from $(u, v) Q (z, w)$]. Now, $x \leq u$ and $u \leq z$ give us $x \leq z$, and we have $y \leq w$ from $y \leq v$ and $v \leq w$. By the definition of Q , this spells out $(x, y) Q (z, w)$ as required.

LH2. Suppose R_1 and R_2 are partial orderings on a set X . Show that $R_1 \cap R_2$ is a partial ordering on X .

Solution. Write R for $R_1 \cap R_2$. Assuming R_1 and R_2 are partial orderings, we have to show that R is also a partial ordering. To this end, one shows that R is reflexive, transitive, and antisymmetric.

Reflexivity: For $x \in X$, we have $x R_1 x$ and $x R_2 x$ because both R_1 and R_2 , being partial orderings, are reflexive. In other words, $(x, x) \in R_1$ and $(x, x) \in R_2$. Hence $(x, x) \in R_1 \cap R_2 = R$.

Transitivity: Suppose $x, y, z \in X$ and assume $x R y$ and $y R z$. This means that $(x, y), (y, z) \in R_1$ and $(x, y), (y, z) \in R_2$. Both R_1 and R_2 are transitive, as they are partial orderings. So we have $(x, z) \in R_1$ and $(x, z) \in R_2$. Therefore $(x, z) \in R_1 \cap R_2 = R$. This shows the transitivity of R .

Antisymmetry: If one has both $x R y$ and $y R x$ for $x, y \in X$, then it follows that $x R_1 y$ and $y R_1 x$, as $R \subseteq R_1$. Since R_1 is antisymmetric as befits a partial ordering, we get $x = y$ as required.