IT-math F2003 : Selected Solution(s)
Episode 7, March 18, 2003

FP3. Determine if $\{ \{n\} \mid n \in \mathbb{N} \}$ is a partition of $\mathbb{N}$.

Solution. Call the set in the statement of the exercise P. We show that P is a partition of $\mathbb{N}$. In order to do that, we have to verify that the three properties in the definition of a partition hold for X.

(i) Show that $\bigcup P \overset{\text{def}}{=} \{ x \mid x \in C \in P \text{ for some } C \} = \mathbb{N}$.

If $n \in \mathbb{N}$, then we have $n \in \{n\} \in P$, so $\mathbb{N} \subseteq \bigcup P$. Conversely, if $x \in \bigcup P$, then there is $C \in P$ with $x \in C$. As, by the definition of P, any $C \in P$ is equal to $\{n\}$ for some $n \in \mathbb{N}$, we have that $x \in \{n\}$ for some $n \in \mathbb{N}$. Therefore $x = n \in \mathbb{N}$. Thus $\bigcup P \subseteq \mathbb{N}$. As we have both $\mathbb{N} \subseteq \bigcup P$ and $\bigcup P \subseteq \mathbb{N}$, conclude $\bigcup P = \mathbb{N}$.

(ii) $C \neq \varnothing$ for all $C \in P$.

Since for any $C \in P$ one has $C = \{n\}$ for a natural number n, $C \neq \varnothing$, as singleton sets are non-empty.

(iii) If $C_1, C_2 \in P$ are such that $C_1 \cap C_2 \neq \varnothing$, then $C_1 = C_2$.

Suppose $C_1 \cap C_2 \neq \varnothing$ for some $C_1, C_2 \in P$. By the definition of P, we have $C_1 = \{n_1\}$ and $C_2 = \{n_2\}$ for some $n_1, n_2 \in \mathbb{N}$. If $x \in \{n_1\} \cap \{n_2\}$, then $x \in \{n_1\}$ and $x \in \{n_2\}$. Therefore $x = n_1$ and $x = n_2$. Hence $n_1 = n_2$, so $\{n_1\} = \{n_2\}$, so $C_1 = C_2$ as required.

SC1. Given an equivalence relaton E on a set X, recall the set

$$X/E = \{ [x]_E \mid x \in X \}, \qquad \text{where} \quad [x]_E = \{ y \in X \mid y \; E \; x \}.$$

Write down a careful proof that X/E is a partition of X.

Solution. As in FP3, we verify that X/E has the three properties.

(i) $\bigcup X/E \overset{\text{def}}{=} \{ x \mid x \in C \in X/E \text{ for some } C \} = X$.

For $x \in X$, we have $x \in [x]_E \in X/E$, so $X \subseteq \bigcup X/E$. Since the elements of X/E are equivalence classes of E, and each equivalence class is a subset of X, we have that $\bigcup X/E$, being a union of subsets of X, is itself a subset of X. [In greater detail, if $x \in C \in X/E$ then $C = [y]_E$ for some $y \in X$, so $(x, y) \in E \subseteq X \times X$. Therefore $x \in X$.] Thus $\bigcup X/E \subseteq X$. Since we have inclusion in both directions, we have $\bigcup X/E = X$.

(ii) $C \neq \varnothing$ for all $C \in X/E$.

If $C \in X/E$ then by the definition of X/E one has $C = [x]_E$ for some $x \in X$. Since $x \in [x]_E$, one has $C = [x]_E \neq \varnothing$.

(iii) If $C_1, C_2 \in X/E$ are such that $C_1 \cap C_2 \neq \varnothing$, then $C_1 = C_2$.

Suppose $C_1, C_2 \in X/E$ are such that $C_1 \cap C_2 \neq \varnothing$. We know $C_1 = [x_1]_E$ and $C_2 = [x_2]_E$ for some $x_1, x_2 \in X$. If $y \in [x_1]_E \cap [x_2]_E$ then $x_1 \; E \; y \; E \; x_2$. Hence $x_1 \; E \; x_2$ for E, being an equivalence relation, is transitive. Let us show $[x_1]_E = [x_2]_E$. Suppose $z \in [x_1]_E$. Then $z \; E \; x_1$. Since $x_1 \; E \; x_2$, we also have $z \; E \; x_2$, or, in other words, $z \in [x_2]_E$. This shows $[x_1]_E \subseteq [x_2]_E$. The opposite inclusion $[x_2]_E \subseteq [x_1]_E$ is shown by a totally similar argument (just interchange the roles of x_1 and x_2). Hence $C_1 = [x_1]_E = [x_2]_E = C_2$ as required.

SC2. Suppose the relation R on a set X is a partial ordering as well as an equivalence relation. Show that for $x, y \in X$, one has $x \; R \; y$ iff $x = y$.

Solution. Let $x, y \in X$. If $x = y$ then $x \mathrel{R} y$ because R, being an equivalence relation, is reflexive.

Suppose now $x \mathrel{R} y$. Since R is an equivalence relation, it is symmetric, so we also have $y \mathrel{R} x$. Since R is a partial ordering, R is antisymmetric, so $x \mathrel{R} y$ and $y \mathrel{R} x$ together imply $x = y$ as required.

SC3. Suppose E is an equivalence relation on a set X with $|X| = 12$, and E has two (distinct) equivalence classes each containing 5 elements, and an equivalence class with two elements. Calculate $|E|$, the total number of pairs in E.

Solution. The partition P of X corresponding to the equivalence relation E consists of the three equivalence classes described in the exercise: $P = \{C_1, C_2, C_3\}$, where $|C_1| = |C_2| = 5$ and $|C_3| = 2$. We have

$$E = \{ (x, y) \mid x, y \in C \text{ for some } C \in P \}.$$

Since we already know all elements of P, we can write this as

$$E = \{ (x, y) \mid x, y \in C_1 \} \cup \{ (x, y) \mid x, y \in C_2 \} \cup \{ (x, y) \mid x, y \in C_3 \}$$

$$= (C_1 \times C_1) \cup (C_2 \times C_2) \cup (C_3 \times C_3).$$

For $i \neq j$, we have $C_i \cap C_j = \varnothing$, therefore $C_i \times C_i \cap C_j \times C_j = \varnothing$. So

$$|E| = |C_1 \times C_1| + |C_2 \times C_2| + |C_3 \times C_3| = |C_1|^2 + |C_2|^2 + |C_3|^2 = 5^2 + 5^2 + 2^2 = 54.$$

LH1. Let the relation R on $\mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ be defined by

$$(m, n) \mathrel{R} (m', n') \iff m \cdot n' = m' \cdot n.$$

Is R an equivalence relation?

Solution. We show that R is an equivalence relation by verifying that it is reflexive, symmetric and transitive.

Reflexivity: If $(m, n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ then $m \cdot n = m \cdot n$, so $(m, n) \mathrel{R} (m, n)$.

Symmetry: Suppose $(m, n), (m', n') \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ and $(m, n) \mathrel{R} (m', n')$. This means $m \cdot n' = m' \cdot n$. It follows that $m' \cdot n = m \cdot n'$. Hence $(m', n') \mathrel{R} (m, n)$.

Transitivity: Suppose $(m, n), (m', n'), (m'', n'') \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ and $(m, n) \mathrel{R} (m', n') \mathrel{R} (m'', n'')$. In other words, $m \cdot n' = m' \cdot n$ and $m' \cdot n'' = m'' \cdot n'$. Let us show $(m, n) \mathrel{R} (m'', n'')$.

Case 1. $n = 0$. We then have $m \cdot n' = m' \cdot n = 0$. As $(m, n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$, it follows that $m \neq 0$. Therefore $n' = 0$ for $m \cdot n' = 0$. Analogously, we get $n'' = 0$ from $m' \cdot n'' = m'' \cdot n'$ and $m' \neq 0$. Therfore $m \cdot n'' = 0 = m'' \cdot n$ which spells out $(m, n) \mathrel{R} (m'', n'')$.

Case 2. $n \neq 0$. From $m \cdot n' = m' \cdot n$ it follows that $m' = \frac{m \cdot n'}{n}$ (we are dividing by a non-zero number). Substituting into $m' \cdot n'' = m'' \cdot n'$, we get $\frac{m \cdot n'}{n} \cdot n'' = m'' \cdot n'$. If it was the case that $n' = 0$, then $m' \neq 0$ (for otherwise $(m', n') = (0, 0) \notin \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$). But then $0 = m \cdot n' = m' \cdot n \neq 0$, a contradiction! Threfore $n' \neq 0$ (this is the same argument we ran in Case 1). Hence we can divide both sides of $\frac{m \cdot n'}{n} \cdot n'' = m'' \cdot n'$ by n' to obtain $\frac{m}{n} \cdot n'' = m''$. Multiplying both sides by n, we get $m \cdot n'' = m'' \cdot n$, or, equivalently, $(m, n) \mathrel{R} (m'', n'')$ as required.

LH2. Let X be a set and $Y \subseteq X$. Define a relation R on $\mathcal{P}(X)$ by

$$A \mathrel{R} B \iff A \triangle B \subseteq Y.$$

(Recall that $\triangle$ is the symmetric difference: $A \triangle B = (A \cup B) \setminus (A \cap B)$.) Show that R is an equivalence relation on $\mathcal{P}(X)$.

Solution. Reflexivity: $A \vartriangle A = \varnothing \subseteq Y$, so $A\ R\ A$ for all $A \in \mathcal{P}X$.

Symmetry: If $A\ R\ B$ then $A \vartriangle B \subseteq Y$. Also,

$$B \vartriangle A = (B \cup A) \setminus (B \cap A) = (A \cup B) \setminus (A \cap B) = A \vartriangle B \subseteq Y,$$

so $B\ R\ A$.

Transitivity: Suppose $A\ R\ B\ R\ C$, i.e. $A \vartriangle B, B \vartriangle C \subseteq Y$. Hence $(A \vartriangle B) \cap (B \vartriangle C) \subseteq Y$. We are going to show that $A \vartriangle C \subseteq (A \vartriangle B) \cap (B \vartriangle C)$, for then iy will follow that $A \vartriangle C \subseteq Y$, so $A\ R\ C$.

If $x \in A \vartriangle C = (A \cup C) \setminus (A \cap C)$ then $x \in A \cup C$ and $x \notin A \cap C$. Suppose $x \in A$. Then $x \notin C$.

Case 1. $x \in B$.
Then $x \in B \vartriangle C \subseteq (A \vartriangle B) \cap (B \vartriangle C)$.

Case 2. $x \notin B$.
Then $x \in A \vartriangle B \subseteq (A \vartriangle B) \cap (B \vartriangle C)$.

Thus in both cases $x \in (A \vartriangle B) \cap (B \vartriangle C)$. If $x \in C$ (so $x \notin A$), an entirely similar argument also shows $x \in (A \vartriangle B) \cap (B \vartriangle C)$. Therefore $A \vartriangle C \subseteq (A \vartriangle B) \cap (B \vartriangle C)$, which by the preceding discussion concludes the solution.

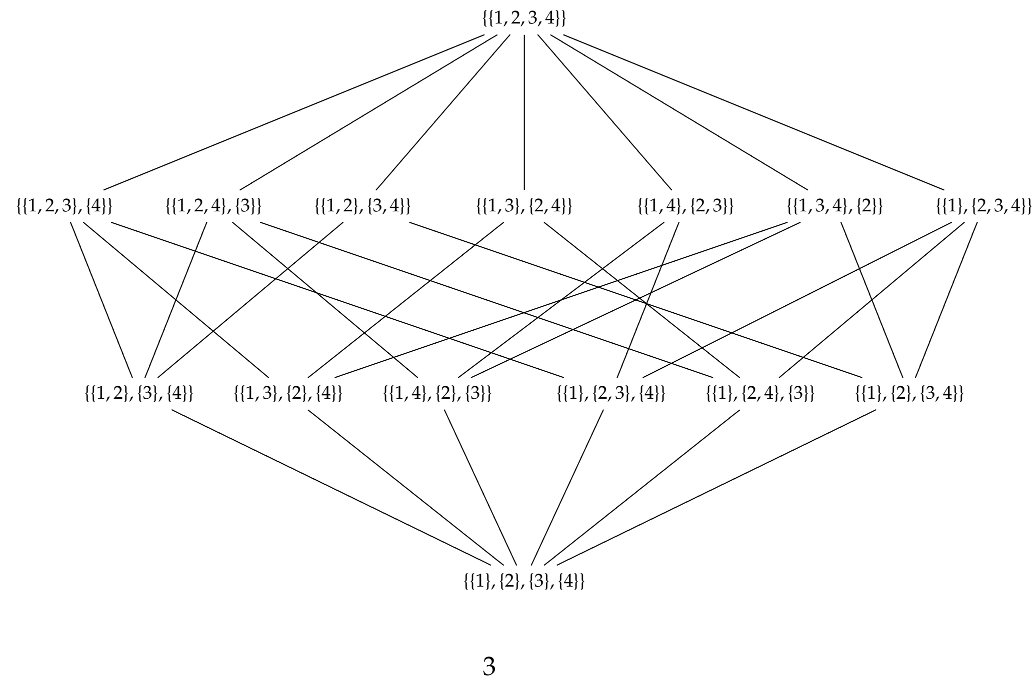
LH3. Suppose $|X| = 4$. Consider the set $\mathcal{E}$ of all equivalence realtions on X. This set is partially ordered by $\subseteq$. Draw the Hasse diagram of the poset $(\mathcal{E}, \subseteq)$.

Solution. For simplicity, let us denote the four elements of the set X by $1, 2, 3, 4$.

For two equivalence relations E_1, E_2 on X, we have $E_1 \subseteq E_2$ if and only if each equivalence class of E_1 is a subset of some equivalence class of E_2: Indeed, suppose each equivalence class of E_1 is a subset of some equivalence class of E_2. If $x\ E_1\ y$ then $[x]_{E_1} = [y]_{E_1} \subseteq [z]_{E_2}$ for some $z \in X$, so $x\ E_2\ z\ E_2\ y$. Thus $E_1 \subseteq E_2$ Conversely, suppose $E_1 \subseteq E_2$. If $y \in [x]_{E_1}$ then $(y, x) \in E_1 \subseteq E_2$ so $y\ E_2\ x$. Hence $y \in [x]_{E_2}$. Thus $[x]_{E_1} \subseteq [x]_{E_2}$.

Recalling that the equivalence classes of an equivalence relation are exactly the components of the corresponding partition, we can reformulate our observation by saying that $E_1 \subseteq E_2$ iff every component of P_1 is a subset of some component of P_2, where P_i is the partition of X corresponding to E_i.

This allows us to find out whether $E_1 \subseteq E_2$ by looking at the partitions corresponding to E_i, and instead of equivalence relations themselves, we write the corresponding partitions (that generally have much shorter descriptions) as the nodes of the Hasse diagram of $(\mathcal{E}, \subseteq)$. So here is the diagram:



DS1(b). For $n, k \in \mathbb{N}_+$ with $k \leq n$, let $\left\{ {n \atop k} \right\}$ be the number of equivalence relations on a set with n elements having exactly k equivalence classes (these are known as *Stirling numbers of the second kind*). For $1 \leq k < n$, show that $\left\{ {n+1 \atop k+1} \right\} = \left\{ {n \atop k} \right\} + (k+1) \cdot \left\{ {n \atop k+1} \right\}$.

Solution. Let $\mathcal{E}_l^m$ denote the set of all equivalence relations on $\{1, \ldots, m\}$ with exactly l many equivalence classes.

Let $\mathcal{A} \subseteq \mathcal{E}_{k+1}^{n+1}$ be the subset of all those equivalence relations E on $\{1, \ldots, n+1\}$ s.t. the number $n+1$ is all alone in its E-equivalence class, i.e. $[n+1]_E = \{n+1\}$. Let $\mathcal{J} \subseteq \mathcal{E}_{k+1}^{n+1}$ be the subset of those equivalence relations E for which $[n+1]_E \neq \{n+1\}$, i.e. for which there is a positive integer $m \leq n$ with $m \ E \ n$. Clearly, $\{\mathcal{A}, \mathcal{J}\}$ is a partition of $\mathcal{E}_{k+1}^{n+1}$, so $|\mathcal{E}_{k+1}^{n+1}| = |\mathcal{A}| + |\mathcal{J}|$.

For an equivalence relation $E \in \mathcal{E}_{k+1}^{n+1}$, define $\tilde{E}$ to be the equivalence relation on $\{1, \ldots, n\}$ obtained 'by throwing away $n+1$': for $i, j \in \{1, \ldots, n\}$, put $i \ \tilde{E} \ j \iff i \ E \ j$.

Given $E \in \mathcal{A}$, we have $\tilde{E} \in \mathcal{E}_k^n$, as the departure of $n+1$ reduces the number of equivalence classes by 1. Given $D \in \mathcal{E}_k^n$, there is exactly one equivalence relation $E \in \mathcal{E}_{k+1}^{n+1}$ with $\tilde{E} = D$, namely $E = D \cup \{(n+1, n+1)\}$. Therefore, there are exactly as many equivalence relations in $\mathcal{A}$ as in $\mathcal{E}_k^n$, that is, $|\mathcal{A}| = |\mathcal{E}_k^n|$.

Given $E \in \mathcal{J}$, we have $\tilde{E} \in \mathcal{E}_{k+1}^n$, because there is an $m < n+1$ with $m \ E \ n+1$, and so the disappearance of $n+1$ does not reduce the number of equivalence classes: $\varnothing \neq [m]_{\tilde{E}} \subseteq [m]_E = [n+1]_E$. Given $D \in \mathcal{E}_{k+1}^n$, we are going to show that there are exactly $k+1$ equivalence relations in $\mathcal{J}$ with $\tilde{E} = D$. Since in any such E, the number $n+1$ has to share an equivalence class with some of the numbers in $\{1, \ldots, n\}$, the relation E is completely determined by pointing out the equivalence class C of D such that $[n+1]_E \supseteq C$. As there are exactly $k+1$ equivalence classes in D, there are exactly $k+1$ ways to make this choice, therefore there are exactly $k+1$ equivalence relations $E \in \mathcal{J}$ with $\tilde{E} = D$. Thus each equivalence relation in $\mathcal{E}_{k+1}^n$ accounts for exacly $k+1$ equivalence relations in $\mathcal{J}$, and each equivalence relation $E \in \mathcal{J}$ is accounted for by exactly one equivalence relation in $\mathcal{E}_{k+1}^n$, namely, by $\tilde{E}$. Therefore there are exactly $k+1$ times as many equivalence relations in $\mathcal{J}$ as there are in $\mathcal{E}_{k+1}^n$, so $|\mathcal{J}| = (k+1) \cdot |\mathcal{E}_{k+1}^n|$.

Finally, we conclude

$$\left\{ {n+1 \atop k+1} \right\} = |\mathcal{E}_{k+1}^{n+1}| = |\mathcal{A}| + |\mathcal{J}| = |\mathcal{E}_k^n| + (k+1) \cdot |\mathcal{E}_{k+1}^n| = \left\{ {n \atop k} \right\} + (k+1) \cdot \left\{ {n \atop k+1} \right\}$$

as required.