IT-math F2003 : Selected Solution(s)
Episode 8, March 25, 2003

FP1(b). Find out if $\left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, x^2 = y^2 \right\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

Solution. This relation (call it R) fails to be a function, since for, say, $x = 1$ there are $y_1 = 1$ and $y_2 = -1$ (observe that $y_1 \neq y_2$) with $(x, y_1), (x, y_2) \in R$.

FP2(b). Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$ is one-to-one and/or onto.

Solution. This function is not one-to-one, for we have $f(-1) = 1 = f(1)$, and $-1 \neq 1$.

Neither is it onto: the square of any real number is non-negative, so $-1 \notin \operatorname{rng} f$.

FP2(c). Determine whether the function $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n + 2$ is one-to-one and/or onto.

Solution. This function is one-to-one, for if $f(n_1) = f(n_2)$, then $n_1 + 2 = n_2 + 2$, so $n_1 = n_2$.

It is, however, not onto: there is no $n \in \mathbb{N}$ with $f(n) = n + 2 = 1$ (recall that $-1 \notin \mathbb{N}$), so $1 \notin \operatorname{rng} f$.

SC1. Show that every set of 15 socks chosen from among 13 pairs of socks contains at least one matching pair.

Solution. Let C be the set of 15 socks that are chosen. Let P be the set of the 13 pairs of socks featured in the statement of the exercise. (Observe that elements of the set P are *pairs* of socks rather than individual socks, so that $|P| = 13$.) Further, consider the obvious function $f : C \to P$ given by

$$f(s) = \text{the pair that the sock } s \text{ comes from.}$$

Since $|C| = 15 > 13 = |P|$, the Pigeonhole Principle says that there must be $s_1, s_2 \in C$ with $s_1 \neq s_2$ and $f(s_1) = f(s_2)$, so that s_1 and s_2 come from the same pair. Thus s_1 and s_2 form the required matching pair.

SC2. Show that among any six integers in $\{1, \ldots, 10\}$ you can find two whose sum is equal to 11.

Solution. This is not much different from SC1. Let $X = \{1, \ldots, 10\}$, and $Y = \{\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}\}$. Observe that Y is a partition of X, because for any $x \in X$ there is exactly one $C \in Y$ with $x \in C$. Further, it is seen by inspection that $x_1 + x_2 = 11$ for any $x_1 \neq x_2$ coming from the same block (aka component) of the partition Y. Now, let $Z \subseteq X$ be the any six-element subset of X. By the above discussion, for any $z \in Z$ there is exactly one $C \in Y$ with $z \in C$, so we have a function $f : Z \to Y$ defined by

$$f(z) = C \iff z \in C.$$

Since $|Z| = 6 > 5 = |Y|$, the Pigeonhole Principle tells us that there are $z_1, z_2 \in Z$ with $z_1 \neq z_2$ and $f(z_1) = f(z_2)$. Thus z_1 and z_2 come from the same block of Y, hence $z_1 + z_2 = 11$.

SC3. Give an example of an onto function from $\mathbb{R}$ to $\mathbb{Z}$.

Solution. Put $f(x) = \lfloor x \rfloor$. This is defined for any real x and always gives an integer, so f is a function from $\mathbb{R}$ to $\mathbb{Z}$. To see that it is onto, take an arbitrary $n \in \mathbb{Z}$. Since $\mathbb{Z} \subseteq \mathbb{R}$, we also have $n \in \mathbb{R}$, so we can apply f to n. We have $f(n) = \lfloor n \rfloor = n$, so $n \in \operatorname{rng} f$. Thus f is onto.

LH1. Let $a_1, \dots, a_n \in \mathbb{Z}$. Show that there are i, j with $1 \le i \le j \le n$ such that $\sum_{k=i}^{j} a_k$ is divisible by n.

Solution. Define $f : \{0, \dots, n\} \to \{0, \dots, n-1\}$ by

$$f(m) = \sum_{k=1}^{m} a_k \bmod n.$$

(One puts $\sum_{k=1}^{0} a_k = 0$, as here one takes the sum of zero many summands.) By the Pigeonhole Principle, there are $i, j \in \{0, \dots, n\}$ with $f(i) = f(j)$ and $i \ne j$ (so we might as well assume $i < j$, renaming the two numbers if necessary). Now, $f(i) = f(j)$ tells us that

$$f(i) = \sum_{k=1}^{i} a_k \bmod n = \sum_{k=1}^{j} a_k \bmod n = f(j),$$

so

$$\sum_{k=i+1}^{j} a_k = \sum_{k=1}^{j} a_k - \sum_{k=1}^{i} a_k \equiv 0 \pmod{n},$$

or, in other words, n divides $\sum_{k=i+1}^{j} a_k$. Since $1 \le i+1 \le j \le n$, we are done.

LH2. Let E be an equivalence relation on X. Suppose the function $\mu : X/E \to X$ is such that $\mu(C) \in C$ for all $C \in X/E$. Show that μ is one-to-one.

Solution. Let $C_1, C_2 \in X/E$, and suppose $\mu(C_1) = \mu(C_2)$. By the condition in the statement of the exercise we have $\mu(C_1) \in C_1$ and $\mu(C_2) \in C_2$. Therefore $\mu(C_1) = \mu(C_2) \in C_1 \cap C_2$. Since C_1 and C_2 are equivalence classes of E and their intersection is non-empty, we must have $C_1 = C_2$. This shows that μ is one-to-one.

LH3. Let $m, n \in \mathbb{N}_+$, and consider the function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $f([x]_n) = [m]_n \cdot [x]_n$. Show that (i) f is one-to-one if and only if (ii) f is onto if and only if (iii) $\gcd(m, n) = 1$.

Solution. Recall the following theorem from the lectures (or Theorem 2.3.3 in the textbook): $\gcd(m, n) = 1$ iff $[m]_n$ has a (multiplicative) inverse in $\mathbb{Z}_n$ iff $[m]_n$ is not a zero-divisor in $\mathbb{Z}_n$.

(i)$\Longrightarrow$(iii): We reason by contraposition. Suppose $\gcd(m, n) \ne 1$. Then by the Theorem $[m]_n$ is a zero-divisor in $\mathbb{Z}_n$. That is, there must exist $[k]_n \ne [0]_n$ with $[k]_n \cdot [m]_n = [0]_n$. But then $f([k]_n) = [0]_n = f([0]_n)$, so f fails to be one-to-one.

(ii)$\Longrightarrow$(iii): If f is onto then $[m]_n \cdot [k]_n = f([k]_n) = [1]_n$ for some $[k]_n \in \mathbb{Z}_n$. In other words, $[k]_n$ is a multiplicative inverse to $[m]_n$. By the Theorem, this implies that $\gcd(m, n) = 1$.

(iii)$\Longrightarrow$(i), (ii): If $\gcd(m, n) = 1$ then by the Theorem, $[m]_n$ has a multiplicative inverse in $\mathbb{Z}_n$, say $[k]_n \cdot [m]_n = [1]_n$. Suppose $f([r_1]_n) = f([r_2]_n)$, or, equivalently, $[m]_n \cdot [r_1]_n = [m]_n \cdot [r_2]_n$. Multiply both parts by $[k]_n$ to get

$$[r_1]_n = [1]_n \cdot [r_1]_n = [k]_n \cdot [m]_n \cdot [r_1]_n = [k]_n \cdot [m]_n \cdot [r_1]_n = [1]_n \cdot [r_2]_n = [r_2]_n,$$

which shows that f is one-to-one.

As for f being onto, given $[r]_n \in \mathbb{Z}_n$, compute

$$f([k \cdot r]_n) = [m]_n \cdot [k]_n \cdot [r]_n = [1]_n \cdot [r]_n = [r]_n,$$

so that $[r]_n \in \operatorname{rng} f$, which shows that f is onto.

DS1. Let X be an arbitrary 8-element subset of $\{1, \ldots, 20\}$. Show that there are two distinct 3-element subsets S and T of X such that the sum of elements of S is equal to that of elements of T.

Solution. Let $[X]^3 = \{T \subseteq X \mid |X| = 3\}$ be the set of all 3-element subsets of X. For $T \in [X]^3$, define $\Sigma(T)$ to be the sum of T's three elements. Since $T \subseteq \{1, \ldots, 20\}$, we have that $6 = 1+2+3 \leq \Sigma(T) \leq 18+19+20 = 57$. Therefore Σ is a function from $[X]^3$ to $Z = \{6, \ldots, 57\}$. We have $|[X]^3| = \binom{8}{3} = 56 > 52 = |Z|$, therefore by the Pigeonhole Principle there are $S, T \in X$ with $\Sigma(S) = \Sigma(T)$ and $S \neq T$ as required.