IT-math F2003 : Selected Solution(s)
Episode 9, April 1, 2003

SC1. If $f : X \to Y$ and $g : Y \to Z$ are bijections, show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Solution. Let z be an arbitrary element of Z. We are going to show that $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$ which suffices to show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

We have

$$(g \circ f)((f^{-1} \circ g)^{-1}(z)) = (g \circ f \circ f^{-1} \circ g^{-1})(z) = (g \circ I_Y \circ g^{-1})(z) = (g \circ g^{-1})(z) = I_Z(z) = z$$

(where $I_W : W \to W$ is the identity function: $I_W(w) = w$ for all $w \in W$). Therefore we have shown $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$ as we said we would.

SC2. Given the function $f = \{(a, b), (b, a), (c, b)\}$ from $X = \{a, b, c\}$ to X (a, b, c all distinct), write $f \circ f$ and $f \circ f \circ f$ as sets of pairs. Further, define $f^{(1)} = f$, $f^{(n+1)} = f \circ f^{(n)}$, and write $f^{(678)}$ as a set of pairs.

Solution. We have $f(a) = b$, $f(b) = a$, and $f(c) = b$. Therefore $f \circ f(a) = f(b) = a$, $f \circ f(b) = f(a) = b$, and $f \circ f(c) = f(b) = a$. Thus $f \circ f = \{(a, a), (b, b), (c, a)\}$. Similarly, $f \circ f \circ f = f^{(3)} = \{(a, b), (b, a), (c, b)\}$.

Writing out a few more iterates, it is easy to observe that, for even $k > 0$ one has $f^{(k)} = \{(a, a), (b, b), (c, a)\}$. The following is then readily proved by induction: If $i \geq 1$ then $f^{(2i)} = \{(a, a), (b, b), (c, a)\}$. Indeed, we have alredy seen this for $i = 1$ (recall that $f^{(2)} = f \circ f$). If we assume $f^{(2i)} = \{(a, a), (b, b), (c, a)\}$ then $f^{(2(i+1))} = f \circ f \circ f^{(2i)}$, so $f^{(2(i+1))}(a) = (f \circ f)(f^{(2i)}(a)) = (f \circ f)(a) = a$, and similarly $f^{(2(i+1))}(b) = b$, and $f^{(2(i+1))}(c) = a$, which completes the induction step. Thus in particular $f^{(678)} = f^{(2 \cdot 339)} = \{(a, a), (b, b), (c, a)\}$.

SC3. Construct a bijection between $\mathbb{N}$ and $\{n \in \mathbb{N} \mid n \text{ odd}\}$.

Solution. Let us call the set $\{n \in \mathbb{N} \mid n \text{ odd}\}$ by the name O. We define a function $f : \mathbb{N} \to O$ putting $f(n) = 2n + 1$. To show that f is a bijection, we establish that it is both one-to-one and onto.

If $f(n_1) = f(n_2)$ then $2n_1 + 1 = 2n_2 + 1$, hence $n_1 = n_2$. Therefore f is one-to-one.

Given an odd natural number m, let us divide it with remainder by 2: $m = 2k + 1$ (the remainder has to be equal to 1, for m is odd). Also, since $m \geq 0$, we have that k is a non-negative integer, or in other words a natural number. Clearly, $f(k) = m$. Thus f is onto.

LH1. Let X and Y be non-empty subsets of $\mathbb{N}$. Prove that there exists a 1–1 function from X to Y if and only if there exists an onto function from Y to X.

Solution. (if). Suppose $g : Y \to X$ is onto. Let us construct a 1–1 function $f : X \to Y$. Given $x \in X$, put $f(x)$ equal to the smallest element in the set $g^{-1}[x] = \{ y \in Y \mid g(y) = x \}$. Since g is onto, we know this set is non-empty, and the smallest element in any non-empty set of natural numbers exists by one of the forms of the Induction Principle. The function f has to be 1–1 because if $f(x_1) = f(x_2)$ then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$.

(only if). Suppose $f : X \to Y$ is 1–1. We construct an onto function $g : Y \to X$. Since X is non-empty we can fix an element $x_0 \in X$. Define

$$g(y) = \begin{cases} x & \text{if } x \in X \text{ is such that } f(x) = y \\ x_0 & \text{if no such } x \in X \text{ exists.} \end{cases}$$

Since f is 1–1, for a given $y \in Y$ there is at most one $x \in X$ with $f(y) = x$, so that the first clause of the definition of $g(y)$ is non-ambiguous. Given $x \in X$, we have $g(f(x)) = x$, so that g is onto as required.

LH2. Let $|X| = n$. Consider the set

$$\mathcal{Z} = \{ (A, B) \mid A \subseteq B \subseteq X \},$$

i.e. the set of all pairs (A, B) of subsets of X with such that the first element of the pair is a subset of the second one. Show that $|\mathcal{Z}| = 3^n$.

Solution. We construct a bijection $F : \mathcal{Z} \to \{0, 1, 2\}^X$. For $(A, B) \in \mathcal{Z}$, $F(A, B)$ has to be a function from X to $\{0, 1, 2\}$, so for $x \in X$ we describe the value $F(A, B)(x)$:

$$F(A,B)(x) = \begin{cases} 0 & \text{if } x \notin B, \\ 1 & \text{if } x \in B \setminus A, \\ 2 & \text{if } x \in A. \end{cases}$$

(so that $F(A, B)(x)$ is the number of sets among A, B that x is an element of). In order to conclude that F is a bijection, we construct the inverse function $G : \{0, 1, 2\}^X \to \mathcal{Z}$ and check that $G \circ F = I_{\mathcal{Z}}$ and $F \circ G = I_{\{0,1,2\}^X}$.

For $f : X \to \{0, 1, 2\}$, define

$$G(f) = (\{ x \in X \mid f(x) = 2 \}, \{ x \in X \mid f(x) \geq 1 \}).$$

For $(A, B) \in \mathcal{Z}$ we have

$$\begin{aligned} G \circ F(A, B) &= (\{ x \in X \mid F(A, B)(x) = 2 \}, \{ x \in X \mid F(A, B)(x) \geq 1 \}) \\ &= (\{ x \in X \mid x \in A \}, \{ x \in X \mid x \in B \}) = (A, B), \end{aligned}$$

so $G \circ F = I_{\mathcal{Z}}$.

For $f : X \to \{0, 1, 2\}$ and $x \in X$ we have

$$\begin{aligned} (F \circ G(f))(x) &= F(\{ x \in X \mid f(x) = 2 \}, \{ x \in X \mid f(x) \geq 1 \})(x) \\ &= \begin{cases} 0 & \text{if } x \notin \{ x \in X \mid f(x) \geq 1 \}, \\ 1 & \text{if } x \in \{ x \in X \mid f(x) \geq 1 \} \setminus \{ x \in X \mid f(x) = 2 \}, \\ 2 & \text{if } x \in \{ x \in X \mid f(x) = 2 \}. \end{cases} \\ &= \begin{cases} 0 & \text{if } f(x) = 0, \\ 1 & \text{if } f(x) = 1, \\ 2 & \text{if } f(x) = 2. \end{cases} \\ &= f(x). \end{aligned}$$

Thus $(F \circ G(f))(x) = f(x)$ for any $x \in X$, hence $F \circ G = I_{\{0,1,2\}^X}$, so we conclude that F is a bijection.

Since we now know there is a bijection between $\mathcal{Z}$ and $\{0, 1, 2\}^X$, we have $|\mathcal{Z}| = |\{0, 1, 2\}^X| = |\{0, 1, 2\}|^{|X|} = 3^n$ as required.

LH3. Prove that a function $f : X \to Y$ is onto if and only if for any set Z and functions $g, h : Y \to Z$ one has that $g \circ f = h \circ f$ implies $g = h$.

Solution. (only if). Suppose $f : X \to Y$ is onto. Let $g, h : Y \to Z$ be such that $g \circ f = h \circ f$. We show $g = h$: Let y be an arbitrary element of Y. Since f is onto, there is an $x \in X$ with $f(x) = y$. But then $g(y) = g(f(x)) = g \circ f(x) = g \circ h(x) = g(h(x)) = g(y)$. Thus for all $y \in Y$ we have $g(y) = h(y)$. Therefore $g = h$.

(if). We reason by contraposition. Suppose $f : X \to Y$ failed to be onto. This means there is a $y_0 \in Y$ such that $f(x) = y_0$ for no $x \in X$. Let $Z = \{0, 1\}$ and define $g(y) = 0$ for all $y \in Y$, $h(y_0) = 1$, and $h(y) = 0$ for all $y \in Y \setminus \{y_0\}$. Observe that $g \neq h$. Yet, given $x \in X$ we have $f(x) \neq y_0$, therefore $h \circ f(x) = h(f(x)) = 0 = g(f(x)) = g \circ f(x)$. Thus $g \circ f = h \circ f$. Summarizing, we have that it is not the case that for any set Z and functions $g, h : Y \to Z$ one has that $g \circ f = h \circ f$ implies $g = h$ (because we have pointed out individual Z, g, h for which one has $g \circ f = h \circ f$ but not $g = h$). This completes the proof by contraposition.