IT-math F2003 : Test Exam Solutions

1. Prove that for each natural number $n \geq 1$ we have

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

Solution. First, we show by induction that $1 + 2 + \cdots + n = \frac{n^2+n}{2}$ for $n \geq 1$. This is clear for $n = 1$, and for induction step we verify

$$1 + 2 + \cdots + n + (n+1) = \frac{n^2+n}{2} + (n+1) = \frac{n^2+n+2n+2}{2} = \frac{(n^2+2n+1)+(n+1)}{2}$$
$$= \frac{(n+1)^2+(n+1)}{2}$$

as required. Next, let us compute

$$(1 + 2 + \cdots + n + (n+1))^2 - (1 + 2 + \cdots + n)^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2 - \left(\frac{n(n+1)}{2}\right)^2$$
$$= \left(\frac{(n+1)(n+2)}{2} - \frac{n(n+1)}{2}\right) \cdot \left(\frac{(n+1)(n+2)}{2} + \frac{n(n+1)}{2}\right)$$
$$= (n+1) \cdot \left(\frac{n+2}{2} - \frac{n}{2}\right) \cdot (n+1) \cdot \left(\frac{n+2}{2} + \frac{n}{2}\right) = (n+1)^2 \cdot (n+1) = (n+1)^3$$

(Here we use the identity $a^2 - b^2 = (a-b)(a+b)$.) We now prove the identity in the problem statement, also by induction on $n \geq 1$. The basis $n = 0$ is clear. Here is the induction step:

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = (1 + 2 + \cdots + n)^2 + (n+1)^3$$
$$= (1+2+\cdots+n)^2 + (1 + 2 + \cdots + n + (n+1))^2 - (1+2+\cdots+n)^2 = (1 + 2 + \cdots + n + (n+1))^2 \,,$$

where the first equality holds by IH. The solution is now complete.

2. Let A and B be finite sets. Show that $|\mathcal{P}(A) \cap \mathcal{P}(B)| = 2^{|A \cap B|}$.

Solution. First, let us observe that $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$: Indeed, for any x we have

$$x \in \mathcal{P}(A) \cap \mathcal{P}(B) \iff x \in \mathcal{P}(A) \text{ and } x \in \mathcal{P}(B) \iff x \subseteq A \text{ and } x \subseteq B$$
$$\iff x \subseteq A \cap B \iff x \in \mathcal{P}(A \cap B).$$

Therefore $|\mathcal{P}(A) \cap \mathcal{P}(B)| = |\mathcal{P}(A \cap B)|$. As $|\mathcal{P}(X)| = 2^{|X|}$ for any finite set X, so for $X = A \cap B$ we get $|\mathcal{P}(A) \cap \mathcal{P}(B)| = |\mathcal{P}(A \cap B)| = 2^{|A \cap B|}$ as desired.

3. Let $R \subseteq X \times X$ be a relation on a non-empty set X. Suppose that R is a partial ordering as well as an equivalence relation on X. Show that R is then also a function from X to X.

Solution. Let us first show that $(x, y) \in R$ if and only if $x = y$ (for all $x, y \in X$). Indeed, if $(x, y) \in R$ then, since R, being an equivalence relation, is symmetric, we also have $(y, x) \in R$. Since R is also antisymmetric (for it is a partial ordering), and $(x, y), (y, x) \in R$, we get $x = y$. Conversely, if $x \in X$ then, since R is an equivalence relation, we have $(x, x) \in R$. Therefore if $x = y$ we do get $(x, y) \in R$.

Thus to each $x \in X$ there is exactly one $y \in X$ such that $(x, y) \in R$ (namely, $y = x$). Therefore, R is a function from X to X. [We also have established that R is the identity function from X to X.]

4. Find integers x and y such that $74x + 118y = 2$.

Solution. Let us apply Euclid's Algorithm to 118 and 74:

$$118 = 1 \cdot 74 + 44$$

$$74 = 1 \cdot 44 + 30$$

$$44 = 1 \cdot 30 + 14$$

$$30 = 2 \cdot 14 + 2$$

$$14 = 7 \cdot 2 + 0$$

[Thus in particular we know that $\gcd(118, 74) = 2$.] Now, substituting backwards we get

$$2 = 30 - 2 \cdot 14 = 30 - 2 \cdot (44 - 30) = 3 \cdot 30 - 2 \cdot 44 = 3 \cdot (74 - 44) - 2 \cdot 44 = 3 \cdot 74 - 5 \cdot 44$$

$$= 3 \cdot 74 - 5 \cdot (118 - 74) = 8 \cdot 74 - 5 \cdot 118.$$

Thus one can take $x = 8$ and $y = -5$.

5. Show that $n^2 + n \cdot (\log_2 n)^{42} = \Theta(n^2)$.

Solution. To show that $n^2 + n \cdot (\log_2 n)^{42} = \Theta(n^2)$ we show that (a) $n^2 + n \cdot (\log_2 n)^{42} = O(n^2)$, and (b) $n^2 + n \cdot (\log_2 n)^{42} = \Omega(n^2)$.

(a). We know that $(\log_2 n)^{42} = o(n)$, hence $(\log_2 n)^{42} = O(n)$. Therefore there is some $C > 0$ such that $(\log_2 n)^{42} \leq C \cdot n$ for almost all n. Hence for almost all n one has $n \cdot (\log_2 n)^{42} \leq C \cdot n^2$. Therefore $n \cdot (\log_2 n)^{42} = O(n^2)$. We also know that $n^2 = O(n^2)$ (take $C = 1$). Since the sum of two functions of which both are $O(n^2)$ is itself $O(n^2)$, we get $n^2 + n \cdot (\log_2 n)^{42} = O(n^2)$ as required.

(b). Here we have for $n \geq 1$ that $|n^2 + n \cdot (\log_2 n)^{42}| = n^2 + n \cdot (\log_2 n)^{42} \geq n^2 = 1 \cdot |n^2|$. Hence $n^2 + n \cdot (\log_2 n)^{42} = \Omega(n^2)$.

6. Construct a context-free grammar G such that

$$\mathcal{L}(G) = \{\, a^n b^n c^k \mid n, k \in \mathbb{N} \,\}.$$

Solution. We take the terminal alphabet $\{a, b, c\}$, the non-terminal alphabet $\{S, A, B\}$ (S is the start symbol), and the following productions:

$$S \to AB, \quad A \to aAb, \quad A \to \varepsilon, \quad B \to cB, \quad B \to \varepsilon.$$

7. Let $\mathcal{R}$ be the set of *all* regular languages over the alphabet $\{a, b\}$. Consider the function $f : \mathcal{R} \to \mathcal{R}$ defined by $f(L) = \{a\}^* \circ L$. Is the function f injective? Motivate your answer.

Solution. The function f is not injective, for consider the languages $\{\varepsilon\}, \{a\}^* \in \mathcal{R}$. We have

$$f(\{\varepsilon\}) = \{\varepsilon\} \circ \{a\}^* = \{a\}^* = \{a\}^* \circ \{a\}^* = f(\{a\}^*),$$

yet $\{\varepsilon\} \neq \{a\}^*$. Thus f is not one-to-one.

8. Given 5 distinct points in the plane, each with integer coordinates, consider the intervals connecting each pair of these points. Show that the middlepoint of at least one of these intervals has integer coordinates.

[Recall that the middlepoint of the interval connecting two points with coordinates (x_1, y_1) and (x_2, y_2) has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.]

Solution. Let the coordinates of the five points be (x_i, y_i) ($i \in \{1, \ldots, 5\}$). Consider the mapping f from the set of the five given points to $\mathbb{Z}_2 \times \mathbb{Z}_2$ defined by $f(x_i, y_i) = ([x_i]_2, [y_i]_2)$. Since there are five points and $|\mathbb{Z}_2 \times \mathbb{Z}_2| = |\mathbb{Z}_2| \cdot |\mathbb{Z}_2| = 2 \cdot 2 = 4 < 5$, by the Pigeonhole Principle there are $i \neq j$ with $i, j \in \{1, \ldots, 5\}$ with $f(x_i, y_i) = f(x_j, y_j)$. This means that $x_i - x_j$ is even, as is $y_i - y_j$. Now, this implies that $x_i + x_j$ is also even (for x_i and x_j are either both even or both odd), and similarly $y_i + y_j$ is even. Hence $\left(\frac{x_i+x_j}{2}, \frac{y_i+y_j}{2}\right)$, the middlepoint of the interval with endpoints (x_i, y_i) and (x_j, y_j), is a pair of integers as was required to show.