Kripke Models over Recursive Worlds
(The Joy of Ultrametric Spaces)

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Plan

- Intro and overview of recent work on using Kripke models over recursively defined worlds
  - to model type systems and logics for dynamically allocated, often higher order, recursive structures (e.g., higher-order store / storable locks)
  - where recursive worlds defined in category of ultrametric spaces
- Talk: partly technical intro to such uses via an example, partly an overview with pointers to literature (more details in other talks by collaborators this week).
- Will focus on the core issue of recursive worlds, using *simple* notions of worlds, both in denotational and operational settings — for useful applications necessary to
  1. use more sophisticated worlds (e.g., for reasoning about local state)
  2. also use recursively defined operations on worlds (e.g., for higher-order frame rules)

  Will skip 1 almost entirely (intro to such in Derek's talk), will only touch briefly upon 2 (more about this in Jan's talk)

Papers can be found at `www.itu.dk/people/birkedal/papers`.

Case Study — Model of $F_{\mu,\text{ref}}$

[BST - FOSSACS'09, MSCS'10]

Slogan: one domain equation for each of $\forall$, ref, μ.

- $\forall$ impredicative polymorphism: choose to model types as relations *UARel*(V) over a recursively defined predomain V.
- ref general references with dynamic allocation: use Kripke model with recursively defined worlds, approximately of the form:

$$\begin{aligned} \mathcal{T} &= \mathcal{W} \rightarrow \mathit{UARel}(V) \\ \mathcal{W} &= \mathbb{N} \rightarrow \mathcal{T} \end{aligned}$$

  Solve in CBUlt.
- μ recursive types: relations interpreting types also recursively defined,
  - non-trivial for reference types, leads to novel modeling of locations involving some approximation information.

Predomain V of values

Proposition. There exists a uniform cpo $(V, (\pi_n)_{n \in \omega})$ satisfying:
In pCpo:

$$V \;\cong\; \mathbb{Z} + Loc + 1 + (V \times V) + (V + V) + V + TV + (V \to TV) \quad (1)$$

where

$$\begin{aligned} TV &= (V \to S \to Ans) \to S \to Ans \\ S &= \mathbb{N} \rightharpoonup_{fin} V \\ Ans &= (\mathbb{Z} + Err)_\perp \end{aligned}$$

and

$$\begin{aligned} Loc &= \mathbb{N} \times \overline{\omega} \\ Err &= 1 \,. \end{aligned}$$

The functions $\pi_n : V \to V_\perp$ satisfy (and are determined by)

$$
\begin{aligned}
\pi_0 &= \lambda v.\perp \\
\pi_{n+1}(in_{\mathbb{Z}}(k)) &= \lfloor in_{\mathbb{Z}}(k) \rfloor \\
\pi_{n+1}(in_{\times}(v_1, v_2)) &= \begin{cases} \lfloor in_{\times}(v_1', v_2') \rfloor & \text{if } \pi_n\, v_1 = \lfloor v_1' \rfloor \text{ and } \pi_n\, v_2 = \lfloor v_2' \rfloor \\ \perp & \text{otherwise} \end{cases} \\
&\ldots \text{etc. as you'd expect, except:} \\
\pi_{n+1}(in_{Loc}(l, m)) &= \lfloor in_{Loc}(l, \min(n+1, m)) \rfloor
\end{aligned}
$$

Untyped Semantics of Terms, I

$[\![t]\!]_X : V^X \to TV$ by induction on t:
Mostly standard, e.g.,

$$[\![\lambda x.\, t]\!]_X\, \rho = \eta(in_{\to}(\lambda v.\ [\![t]\!]_{X,x}\, (\rho[x \mapsto v])))$$

$$[\![t_1\ t_2]\!]_X\, \rho = [\![t_1]\!]_X\, \rho \star \lambda v_1.\ [\![t_2]\!]_X\, \rho \star \lambda v_2. \begin{cases} f\ v_2 & \text{if } v_1 = in_{\to}\, f \\ error & \text{otherwise} \end{cases}$$

Untyped Semantics of Terms, II

- For lookup and assignment we need to consider semantic locations:

$$[\![!\, t]\!]_X\, \rho = [\![t]\!]_X\, \rho \star \lambda v.\, \mathit{lookup}\, v$$

- where $\mathit{lookup}\, v =$

$$\lambda k\, \lambda s. \begin{cases} k\, s(l)\, s & \text{if } v = \lambda_l \text{ and } l \in \mathrm{dom}(s) \\ k\, v'\, s & \text{if } v = \lambda_l^{n+1},\ l \in \mathrm{dom}(s), \text{ and } \pi_n(s(l)) = \lfloor v' \rfloor \\ \bot_{Ans} & \text{if } v = \lambda_l^{n+1},\ l \in \mathrm{dom}(s), \text{ and } \pi_n(s(l)) = \bot \\ \mathit{error}_{Ans} & \text{otherwise} \end{cases}$$

Untyped Semantics of Terms, III

- Adequacy wrt. standard operational semantics can be shown using recursively defined logical relation.
  - Non-trivial, but not too hard using Pitts' technique (with a function-space lattice to deal with nested recursive types), since suffices to consider only *closed* types for adequacy [BST, TLDI'09].
- Now on to typed semantics, i.e., definition of logical relations over the untyped semantics. First define *space of types* using ultrametric spaces.

CBUlt

Recall:

- An *ultrametric space* is a metric space (D, d) that instead of triangle inequality satisfies the stronger *ultrametric inequality*:

$$d(x, z) \leq \max(d(x, y), d(y, z)).$$

- A function $f : D_1 \to D_2$ from a metric space (D_1, d_1) to a metric space (D_2, d_2) is *non-expansive* if $d_2(f(x), f(y)) \leq d_1(x, y)$ for all x and y in D_1.
- A function $f : D_1 \to D_2$ from a metric space (D_1, d_1) to a metric space (D_2, d_2) is *contractive* if there exists $\delta < 1$ such that $d_2(f(x), f(y)) \leq \delta \cdot d_1(x, y)$ for all x and y in D_1.
- CBUlt is the category with complete 1-bounded ultrametric spaces and non-expansive functions.

CBUlt, II

- CBUlt is cartesian closed; the exponential $(D_1, d_1) \to (D_2, d_2)$ is the set of non-expansive maps with the "sup"-metric $d_{D_1 \to D_2}$ as distance function:

$$d_{D_1 \to D_2}(f, g) = \sup\{d_2(f(x), g(x)) \mid x \in D_1\}\,.$$

- Thm [America-Rutten]: Solutions to recursive domain equations for locally contractive functors exist.
- A functor $F : \mathrm{CBUlt}^{\mathrm{op}} \times \mathrm{CBUlt} \to \mathrm{CBUlt}$ is *locally contractive* if there exists $\delta < 1$ such that

$$d(F(f,g), F(f',g')) \leq \delta \cdot \max(d(f,f'), d(g,g'))$$

for all non-expansive functions f, f', g, and g'.

$UARel(V) \in$ CBUlt

Recall [Amadio, Abadi-Plotkin]:

- $UARel(V)$ is the set of admissible relations that are *unifom*: $\varpi_n \in R \to R_\perp$, for all n.
- Such relations are determined by its elements of the form $(\varpi_n\, e, \varpi_n\, e')$.
- $UARel(V) \in$ CBUlt, distance function:

$$d(R,S) = \begin{cases} 2^{-\max\{\, n\in\omega \mid \varpi_n\in R\to S \,\wedge\, \varpi_n\in S\to R \,\}} & \text{if } R \neq S \\ 0 & \text{if } R = S. \end{cases}$$

Worlds

- **Proposition.** Let $(D, d) \in$ CBUlt. The set $\mathbb{N} \rightharpoonup_{fin} D$ with distance function:

$$d'(\Delta, \Delta') = \begin{cases} \max\{d(\Delta(l), \Delta'(l)) \mid l \in \mathrm{dom}(\Delta)\} & \text{if } \mathrm{dom}(\Delta) = \mathrm{dom}(\\ 1 & \text{otherwise.} \end{cases}$$

  is in CBUlt.
- Extension ordering: $\Delta \leq \Delta'$ iff

$$\mathrm{dom}(\Delta) \subseteq \mathrm{dom}(\Delta') \wedge \forall l \in \mathrm{dom}(\Delta).\, \Delta(l) = \Delta'(l)\,.$$

Space of types

- **Proposition.**

$$F(D) = (\mathbb{N} \rightharpoonup_{fin} D) \rightarrow_{mon} UARel(V)$$

  (monotone, non-expansive maps) defines a functor $F : CBUlt^{\mathrm{op}} \rightarrow CBUlt$.

- **Theorem.** There exists $\widehat{\mathcal{T}} \in$ CBUlt such that the isomorphism

$$\widehat{\mathcal{T}} \cong \tfrac{1}{2}((\mathbb{N} \rightharpoonup_{fin} \widehat{\mathcal{T}}) \rightarrow_{mon} UARel(V)) \tag{2}$$

  holds in CBUlt.

Space of Types, II

Define:

- Worlds: $\mathcal{W} = \mathbb{N} \rightharpoonup_{fin} \widehat{\mathcal{T}}$
- Types: $\mathcal{T} = \mathcal{W} \rightarrow_{mon} \mathit{UARel}(V)$
- Computations: $\mathcal{T}_T = \mathcal{W} \rightarrow_{mon} \mathit{UARel}(TV)$
- Continuations: $\mathcal{T}_K = \mathcal{W} \rightarrow_{mon} \mathit{UARel}(K)$
- States: $\mathcal{T}_S = \mathcal{W} \rightarrow \mathit{UARel}(S)$ (note: not monotone)

Semantics of Types

For every $\Xi \vdash \tau$, define the non-expansive $[\![\tau]\!]_\Xi : \mathcal{T}^\Xi \to \mathcal{T}$ by induction on τ:

$$
\begin{aligned}
[\![\alpha]\!]_\Xi\,\varphi &= \varphi(\alpha) \\
[\![\mathrm{int}]\!]_\Xi\,\varphi &= \lambda\Delta.\,\{\,(in_{\mathbb{Z}}\,k, in_{\mathbb{Z}}\,k) \mid k \in \mathbb{Z}\,\} \\
[\![1]\!]_\Xi\,\varphi &= \lambda\Delta.\,\{\,(in_1 *, in_1 *)\,\} \\
[\![\tau_1 \times \tau_2]\!]_\Xi\,\varphi &= [\![\tau_1]\!]_\Xi\,\varphi \times [\![\tau_2]\!]_\Xi\,\varphi \\
[\![0]\!]_\Xi\,\varphi &= \lambda\Delta.\,\emptyset \\
[\![\tau_1 + \tau_2]\!]_\Xi\,\varphi &= [\![\tau_1]\!]_\Xi\,\varphi + [\![\tau_2]\!]_\Xi\,\varphi \\
[\![\mathrm{ref}\,\tau]\!]_\Xi\,\varphi &= \mathit{ref}([\![\tau]\!]_\Xi\,\varphi) \\
[\![\forall\alpha.\tau]\!]_\Xi\,\varphi &= \lambda\Delta.\,\{\,(in_\forall\,c, in_\forall\,c') \mid \forall\nu \in \mathcal{T}.\,(c, c') \in \\
&= \qquad\qquad \mathit{comp}([\![\tau]\!]_{\Xi,\alpha}\,\varphi[\alpha \mapsto \nu])(\Delta)\,\} \\
[\![\mu\alpha.\tau]\!]_\Xi\,\varphi &= \mathit{fix}\left(\lambda\nu.\,\lambda\Delta.\,\{\,(in_\mu\,v, in_\mu\,v') \mid (v, v') \in [\![\tau]\!]_{\Xi,\alpha}\,\varphi[\alpha \mapsto \nu]\,\Delta\,\}\right) \\
[\![\tau_1 \to \tau_2]\!]_\Xi\,\varphi &= ([\![\tau_1]\!]_\Xi\,\varphi) \to (\mathit{comp}([\![\tau_2]\!]_\Xi\,\varphi))
\end{aligned}
$$

Semantic Type Constructors

$$(\nu_1 \times \nu_2)(\Delta) = \{\, (in_\times(v_1, v_2), in_\times(v_1', v_2')) \mid (v_1, v_1') \in \nu_1(\Delta) \wedge (v_2, v_2') \in \nu_2(\Delta) \,\}$$

$$\begin{aligned} ref(\nu)(\Delta) = \{\, (\lambda_l, \lambda_l) \mid l \in \mathrm{dom}(\Delta) \wedge \\ \forall \Delta_1 \geq \Delta.\ App\,(\Delta(l))\,\Delta_1 = \nu(\Delta_1) \,\} \\ \cup \{\, (\lambda_l^{n+1}, \lambda_l^{n+1}) \mid l \in \mathrm{dom}(\Delta) \wedge \\ \forall \Delta_1 \geq \Delta.\ App\,(\Delta(l))\,\Delta_1 \stackrel{n}{=} \nu(\Delta_1) \,\} \end{aligned}$$

- Note the use of semantic locations to ensure non-expansiveness in *ref* case.
- Necessary: see Kristian's talk.
- Because of relational parametricity, we need to model *open* types; hence need to compare semantic types above, cannot simply use syntactic worlds and compare types syntactically.

Semantic Type Constructors, II

$$(\nu \to \xi)(\Delta) = \{\, (in_{\to}\, f, in_{\to}\, f') \mid \forall \Delta_1 \geq \Delta.\ \forall (v, v') \in \nu(\Delta_1)\,.(f\,v, f'\,v') \in \xi(\Delta_1)\,\}$$

$$cont(\nu)(\Delta) = \{\, (k, k') \mid \forall \Delta_1 \geq \Delta. \forall (v, v') \in \nu(\Delta_1).\ \forall (s, s') \in states(\Delta_1).\,(k\,v\,s, k'\,v'\,s') \in R_{Ans}\,\}$$

$$comp(\nu)(\Delta) = \{\, (c, c') \mid \forall \Delta_1 \geq \Delta. \forall (k, k') \in cont(\nu)(\Delta_1).\ \forall (s, s') \in states(\Delta_1).\,(c\,k\,s, c'\,k'\,s') \in R_{Ans}\,\}$$

$$states(\Delta) = \{\, (s, s') \mid \mathrm{dom}(s) = \mathrm{dom}(s') = \mathrm{dom}(\Delta)\ \wedge\ \forall l \in \mathrm{dom}(\Delta).\,(s(l), s'(l)) \in App\,(\Delta(l))\,(\Delta)\,\}$$

$$R_{Ans} = \{\, (\bot, \bot)\,\} \cup \{\, (\lfloor \iota_1\, k \rfloor, \lfloor \iota_1\, k \rfloor) \mid k \in \mathbb{Z}\,\}$$

Typed Semantics of Terms

- For $\Xi \vdash \Gamma$ and $\varphi \in \mathcal{T}^{\Xi}$, let $[\![\Gamma]\!]_{\Xi}\,\varphi$ be the binary relation on $V^{\mathrm{dom}(\Gamma)}$ defined by

$$[\![\Gamma]\!]_{\Xi}\,\varphi = \{\,(\rho, \rho') \mid \forall x \in \mathrm{dom}(\Gamma).\,(\rho(x), \rho'(x)) \in [\![\Gamma(x)]\!]_{\Xi}\,\varphi\,\}\,.$$

- Two typed terms $\Xi \mid \Gamma \vdash t : \tau$ and $\Xi \mid \Gamma \vdash t' : \tau$ of the same type are *semantically related*, written $\Xi \mid \Gamma \models t \sim t' : \tau$, if for all $\varphi \in \mathcal{T}^{\Xi}$, all $(\rho, \rho') \in [\![\Gamma]\!]_{\Xi}\,\varphi$, and all $\Delta \in \mathcal{W}$,

$$\left([\![t]\!]_{\mathrm{dom}(\Gamma)}\,\rho,\ [\![t']\!]_{\mathrm{dom}(\Gamma)}\,\rho'\right) \in \mathit{comp}([\![\tau]\!]_{\Xi}\,\varphi)(\Delta)\,.$$

Typed Semantics of Terms, II

- **Theorem.** Semantic relatedness is a congruence.
- **Corollary.** (FTLR) If $\Xi \mid \Gamma \vdash t : \tau$, then $\Xi \mid \Gamma \models t \sim t : \tau$.
- **Corollary.** If $\emptyset \mid \emptyset \vdash t : \tau$ is a closed term of type τ, then $[\![t]\!]_{\emptyset}\,\emptyset \neq \mathit{error}$.
- **Corollary.** If $\Xi \mid \Gamma \models t \sim t' : \tau$ then $t =_{\mathrm{ctx}} t'$.

Overview of Applications and Extensions

- Four strands of work plus mention couple of other applications

Strand I: Nested Triples and (Anti)Frame Rules

Separation Logic with Nested Hoare Triples for reasoning about stored code (higher-order store) with higher-order frame rules [SBRY-CSL'09]

- Interpretation indexed over Kripke world describing "hidden invariants".
- But invariants are simply predicates (think of frame rule where any predicate can be used as an invariant), so get equation in CBUlt:

$$
\begin{aligned}
Pred &= \tfrac{1}{2}(W \to UAdm(H)) \\
W &\cong Pred
\end{aligned}
$$

Iso $\iota : Pred \to W$.

Strand I, ctd

- For higher-order frame rules: define non-expansive map $\circ : W \times W \to W$, s.t., for all $p, r, w \in W$,

$$\iota^{-1}(p \circ r)(w) \;=\; \iota^{-1}(p)(r \circ w) \ast \iota^{-1}(r)(w) \,.$$

- Intuition:
  - p and r world-dependent invariants
  - world-dependency via application
  - $p \circ r$ is the extension of p with r: first extend r with w, and then apply p to that, in addition to "starring on" $r(w)$.
- Well-defined by Banach: intuitively because the $\circ$ on the right is as an argument, below an unfolding via ι^{-1}.
- Semantics allowed to investigate soundness of various higher-order frame rules (tricky, some formulations are not sound, others are, see paper for examples)

Take-home: Use worlds form metric space to ensure well-definedness (via Banach) of recursive operation on worlds.

Strand I, ctd

Models of Pottier's Anti-frame rule [SYBRS - FOSSACS'10]

- Separation Logic for higher-order store with nested triples and formulation of Pottier's anti-frame rule for hiding invariants in direct style.
- Standard existence theorems for the world equation used did not apply directly, constructed solution by hand.
- (Jan's talk)

Strand II: Step-Indexed Models

(cf. Derek's talk)

- Step-indexed model of $F_{\mu,\text{ref}}$ for reasoning about ctx. equiv., with more refined worlds, allows to prove more example programs equivalent. [ADR-POPL'09]
- Logics (LSLR, LADR) for step-indexed models, to avoid reasoning about steps when reasoning about examples (and a bit of the meta-theory) [DAB-LICS'09, DNRB-POPL'10]
- Worlds as transition systems describing how local state can evolve, studying the influence of different langauge features (first vs. higher-order state, with or without call/cc). Handles all known examples. [DNB-ICFP'10]

Take-home: (1) logics for steps for more high-level reasoning, (2) expressive worlds for more useful models.

Strand III: expressive ultrametric worlds

- Scaling up the denotational approach to recursively defined worlds a la those in LADR. [BST-TR] [Thamsborg dissertation]
  - involves using new form of relations that we call Bohr relations (chain-complete and downwards-closed in left-hand side), capturing ctx. *approximation* (instead of equiv.) [as in step-indexed models]
  - involves solving world equation in category of *preordered* ultrametric spaces
    - The Category-Theoretic Solution to Recursive Metric-Space Equations [BST-TCS'10]. Supporting theory. M-categories. (Jacob's talk.)

Take-home (1) Techniques scale well, (2) Resulting model allows for proofs of examples in the model at same level of abstraction as the LADR logic for step-indexed model.

Strand IV: Step-Indexed Kripke Models over Rec. Worlds

Recent work on showing how the approach applies to operational semantics via step-indexing [BRSSTY]

- arguably simpler than denotational approach, scales well to concurrency
- high-level understanding of step-indexing
  - essence of step-indexing
  - generalizes Hobor et. al.'s Indirection Theory [POPL'10], which is aimed at giving general description of step-indexed models
- has been formalized in Coq

To explain idea, let's consider a simple unary model $F_{\mu,\mathrm{ref}}$.

Uniform Predicates

Idea: replace domain V by the set of Val of operational values

- Uniform predicates:

$$UPred(Val) \;=\; \{p \subseteq \mathbb{N} \times Val \mid \forall (k, v) \in p. \forall j \leq k. (j, v) \in p\}$$

- For $p \in UPred(Val)$ and $k \in \mathbb{N}$, let

$$\overline{p}^k = \{(m, v) \in p \mid m < k\}$$

- Distance:

$$d(p, q) \;=\; \begin{cases} 2^{-\max\{k \mid \overline{p}^k = \overline{q}^k\}} & \text{if } p \neq q \\ 0 & \text{otherwise.} \end{cases}$$

- Lemma $(UPred(Val), d)$ is a well-defined object in CBUlt.

Space of Types

Theorem

There exists $\widehat{\mathcal{T}} \in$ CBUlt *such that*

$$\widehat{\mathcal{T}} \;\cong\; \frac{1}{2} \cdot ((\mathbb{N} \rightharpoonup_{fin} \widehat{\mathcal{T}}) \rightarrow_{mon} \mathit{UPred}(\mathit{Val}))$$

is an iso in CBUlt.

Interpretation of Types, I

Define non-expansive map

$$[\![\Xi \vdash \tau]\!] \;:\; \mathcal{T}^{|\Xi|} \rightarrow \mathcal{T}$$

by induction on τ (only some cases):

$$[\![\Xi \vdash \tau]\!]_\eta : \mathcal{W} \rightarrow_{mon} \mathit{UPred}(\mathit{Val})$$

$$[\![\Xi \vdash 1]\!]_\eta w = \{(k, ()) \mid k \in \mathbb{N}\}$$

$$[\![\Xi \vdash \mathrm{ref}\,\tau]\!]_\eta w = \{(k, l) \mid l \in \mathrm{dom}(w) \wedge \\ \forall w' \sqsupseteq w.i(w(l))(w') \stackrel{k}{=} [\![\Xi \vdash \tau]\!]_\eta w'\}$$

Interpretation of Types, II

$$\llbracket \Xi \vdash \tau \to \tau' \rrbracket_\eta w = \{(k, v) \mid \forall v' \in \mathit{Val}.\ \forall w' \sqsupseteq w.\ \forall i \leq k.\ (i, v') \in \llbracket \Xi \vdash \tau \rrbracket_\eta w' \implies (i, v\, v') \in \mathcal{E}\llbracket \Xi \vdash \tau' \rrbracket_\eta w'\}$$

$$\mathcal{E}\llbracket \Xi \vdash \tau \rrbracket_\eta : \mathcal{W} \to_{mon} \mathit{UPred}(\mathit{Exp})$$

$$\mathcal{E}\llbracket \tau \rrbracket_\eta w = \{(k, t) \mid \forall i \leq k.\ \forall h, h'.\ \forall t'.\ (h :_k w \ \wedge\ (t \mid h) \longmapsto^i (t' \mid h') \ \wedge\ (t \mid h') \text{ irreducible}) \implies (\exists w' \sqsupseteq w.\ \ h' :_{k-i} w' \ \wedge\ (k - i, v) \in \llbracket \tau \rrbracket_\eta w')\}$$

$$h :_k w \iff \forall i < k.\ \mathrm{dom}(h) = \mathrm{dom}(w) \ \wedge\ \forall l \in \mathrm{dom}(w).\ \ (i, h(l)) \in w(l)(w)$$

Interpretation of Types, III

- Recursive Types:

$$[\![\Delta \vdash \mu\alpha.\tau]\!]_\eta = \mathit{fix}(\lambda r.\,\lambda w.\,\{(k, \mathrm{fold}\ t) \mid k > 0 \implies (k-1, v) \in [\![\Delta, \alpha \vdash \tau]\!]_{\eta[\alpha \mapsto r]} w\})$$

- Uses Banach's fixed point theorem.
- Contractiveness ensured by use of $k - 1$.

Well-definedness

- Metric setup tells you what you have to show:
  - non-expansiveness of $[\![\Xi \vdash \tau]\!]$
  - non-expansiveness of $[\![\Xi \vdash \tau]\!]_\eta$
  - contractiveness of map for recursive types.
- Simple calculations.

Example lemma

Lemma

If $s :_k w$ *and* $w \stackrel{n}{=}_{\mathcal{W}} w'$ *and* $k < n$*, then also* $s :_k w'$*.*

Proof.

TS: $\forall j < k.\ \mathrm{dom}(s) = \mathrm{dom}(w') \wedge \forall l \in \mathrm{dom}(w').\ (j, s(l)) \in w'(l)(w')$.
Sps. $k > 0$; then $n > 0$. Let $j < k$. By $w \stackrel{n}{=}_{\mathcal{W}} w'$, we get $\mathrm{dom}(w) = \mathrm{dom}(w') \wedge \forall l \in \mathrm{dom}(w).\ \forall w_0.\ w(l)(w_0) \stackrel{n-1}{=} w'(l)(w_0)$. Since $\mathrm{dom}(s) = \mathrm{dom}(w)$ by the assumption that $s :_k w$ (using $k > 0$), we get $\mathrm{dom}(s) = \mathrm{dom}(w')$. Moreover,

$$w(l)(w) \stackrel{n}{=} w(l)(w') \stackrel{n-1}{=} w'(l)(w')$$

since $w(l)$ is non-expansive, and since $w \stackrel{n}{=}_{\mathcal{W}} w'$. Thus, as $(j, s(l)) \in w(l)(w)$ by assumption, and since $j < k \leq n - 1$, we also get $(j, s(l)) \in w'(l)(w')$, as desired. □

Specialization to Indirection Theory

- Indirection Theory. Hobor et. al. POPL'10
  - General formulation of step-indexed models. Also observe cannot solve world-equation in sets. Instead describe approximate solutions and show how they can be used in many step-indexed models.
- We prove that one can derive an approx. solution a la Indirection Theory from one of our metric equations (see paper for detailed formulation and formal theorems).
- Corollary: applies to all the models described by indirection theory.

Advantages of metric approach

(some propaganda :-))

- Useful guiding framework.
- Supporting theory (e.g., recursive equations when spaces equipped with structure).
- Supports recursively-defined operations on worlds.
- Connection between step-indexing and metric spaces known from start of step-indexing (Appel-McAllester); but useful not to forget the connection!
- Also formalized in Coq [BBKV-TR]

Applications of Operational Approach

- We have given an alternative model of nested Hoare triples based directly on operational semantics with higher-order frame rules.
- Defined a model of Pottier's Capability Calculus, shown soundness of extension with higher-order frame and anti-frame rules.
- Capability Calculus setup: $\mathcal{W} \cong \frac{1}{2}\mathcal{W} \to \mathit{UPred}^{\uparrow}(\mathit{Heap})$.
- Model expresses that capabilities can be understood as separation logic assertions and is used to show soundness of the type system (new result).

Other recent applications

- A Metric Model of Nakano's calculus of Guarded Recursion [BSS - FICS'10]
  - kripke logical relation for adequacy proof defined using family of natural-number indexed relations.
- Separation logic for storable locks [BBS - ongoing].
- Model of type-and-effect system for higher-order store, extending work of Benton, Beringer, Hofmann, Kennedy [TB - ongoing] (again seems to involve a recursively defined operation on the set of worlds, though a quite different one).
- Krishnaswami-Benton: model of reactive programming using ultrametric spaces, see Neel's talk.

Thank You!